

Ultraviolet Sensitivity in Higher Dimensions

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ABSTRACT: We calculate the first three Gilkey-DeWitt (heat-kernel) coefficients, a_0 , a_1 and a_2 , for massive particles having the spins of most physical interest in n dimensions, including the contributions of the ghosts and the fields associated with the appropriate generalized Higgs mechanism. By assembling these into supermultiplets we compute the same coefficients for general supergravity theories, and show that they vanish for many examples. One of the steps of the calculation involves computing these coefficients for massless particles, and our expressions in this case agree with – and extend to more general background spacetimes – earlier calculations, where these exist. Our results give that part of the low-energy effective action which depends most sensitively on the mass of heavy fields once these are integrated out. These results are used in hep-th/0504004 to compute the sensitivity to large masses of the Casimir energy in Ricci-flat 4D compactifications of 6D supergravity.

KEYWORDS: Strings, Branes, Cosmology.

Contents

| | |
|--|-----------|
| 1. Introduction | 1 |
| 2. General One-Loop Results | 3 |
| 2.1 The Gilkey-DeWitt Coefficients | 4 |
| 2.2 Spin 0 | 7 |
| 2.3 Spin 1/2 | 8 |
| 2.4 Spin 1 | 9 |
| 2.5 Antisymmetric Tensors | 12 |
| 2.6 Spin 3/2 | 17 |
| 2.7 Spin 2 | 22 |
| 3. Supergravity Models | 31 |
| 3.1 11D Example | 34 |
| 3.2 10D Examples | 35 |
| 3.2.1 Massive 10D Fields | 38 |
| 3.3 6D Examples | 38 |
| 3.3.1 Massless Multiplets | 40 |
| 3.3.2 Massive Multiplets | 40 |
| 3.4 4D Examples | 41 |
| 3.4.1 Massless Multiplets | 41 |
| 3.4.2 Massive Multiplets | 42 |
| 4. Conclusions | 44 |
| A. Appendix: Gravitini With $\Lambda \neq 0$ | 44 |

1. Introduction

It is an experimental fact that Nature comes to us with many scales, and that we do not need to understand them all at once in order to understand the physics of

any particular scale. Indeed, progress on atomic physics did not have to await a complete theory of nuclei, quarks or any hitherto-undiscovered more microscopic constituents, and this fact arguably is fundamental to the very possibility of making progress in science. This elementary physical fact is reflected in the mathematics used to describe the physical world — quantum field theory — through the calculus of renormalization and effective field theories.

Although low-energy physics is largely insensitive to higher energy scales, it is not completely so. After all, the electronic properties of atoms *do* depend on the total charge and mass of the underlying nucleus. The calculus of renormalization, which has become very well-developed over the last few decades, allows the very efficient calculation of the comparatively few ways in which short-distance high-energy physics can affect the physics of longer wavelengths and lower energies. It does so by identifying the low-energy effective field theory which captures the effects of integrating out high-energy modes, and in particular finding which effective interactions are ‘Ultraviolet (UV) Sensitive’ inasmuch as they are proportional to positive powers of the large energy scale, m , of the particles which have been integrated out [1, 2]. It is this existence of UV sensitive terms in the low-energy effective action which underlies the ‘naturalness’ problems of otherwise-successful theories like the Standard Model, including the problems of the Electroweak Hierarchy or of the Cosmological Constant.

Although the techniques for computing UV sensitive interactions is very highly developed for four-dimensional theories, less has been done to compute such terms in higher-dimensional models. The absence of such higher-dimensional results is becoming more of a hindrance given that extra-dimensional ideas are playing an increasingly prominent role in our understanding of the various hierarchy problems [3, 4, 5]. Fortunately, well-developed heat-kernel techniques exist for computing UV sensitivity for reasonably general geometries [6], and it is the purpose of this paper to use these techniques to provide a systematic calculation of the leading sensitivity to heavy masses (within the one-loop approximation) in higher-dimensional theories.

In order to do so we compute the most UV sensitive contributions which are obtained when massive particles are integrated out at one loop. We calculate the leading heat-kernel coefficients for n spacetime dimensions and for a broad class of particle spins, including most particle types which arise within the higher-dimensional supergravities which are of the most modern interest. Similar heat-kernel calcula-

tions have been performed in the past for massless particles [8, 9, 10], and more recently for certain massive fields in 4D [11]. The results presented herein extend these earlier calculations in several ways. Our main extension is to provide the n -dimensional results for massive fields rather than massless ones, including calculating the contributions of the various ghosts and would-be Goldstone particles which participate in the generalized higher-spin mass-acquisition (Higgs) mechanisms. As an intermediate step we also compute the leading heat-kernel coefficients for massless particles, extending previous general results to include a nonzero cosmological constant in n -dimensions. An application of these results to the study of UV sensitivity in Ricci-flat 4D compactifications of 6D supergravity may be found in ref. [12].

Our calculations are presented as follows. The next section, §2, summarizes the general heat-kernel formulae and evaluates them for the various massless fields which arise within higher-dimensional supergravities. These calculations are performed in a covariant gauge, for which the gauge-fixing and ghost contributions are explicitly displayed. For higher-spin fields the generalizations to nonzero masses are computed by coupling the massless fields to the appropriate would-be-Goldstone fields, whose eating makes up the generalized Higgs mechanism for the fields of interest. §3 then applies the general results of §2 to the field content of specific supergravities. As a check, and in order to compare with previous results, the contributions of the massless fields of 10- and 11-dimensional supergravities are computed and shown to sum to zero, in agreement with earlier calculations. The contributions of massive fields and supermultiplets in 4, 6 and 10 dimensions are also tabulated in this section.

2. General One-Loop Results

This section collects the results for the most ultraviolet-sensitive parts of the one-loop action obtained by integrating out massless and massive particles having spins up to and including spin two. For the present purposes we take the one-loop approximation to represent the field theory which is obtained by linearizing the various field equations about a particular background configuration. That is, denoting the set of (real) quantum fields generically by Φ^i , with background value φ^i , we write $\Phi^i = \varphi^i + \phi^i$ and expand the classical action to quadratic order in ϕ^i :

$$S \approx -\frac{1}{2} \int d^n x \phi^i \Delta_{ij}(\varphi) \phi^j. \quad (2.1)$$

Here n denotes the dimension of spacetime, and in practice we consider nonzero backgrounds only for scalar, gauge and gravitational fields. We do, however, allow *fluctuations* about these backgrounds for all of the most commonly encountered fields in higher-dimensional supergravity theories.

2.1 The Gilkey-DeWitt Coefficients

The full one-loop quantum correction to the effective action, Σ , in the presence of various background fields can be explicitly calculated provided one can evaluate the functional determinant of the relevant differential operator in the presence of those backgrounds. For a basis of real fields, ϕ^j , whose linearized equation of motion is $\Delta^i_j \phi^j$ the one loop contribution to the effective action is

$$i\Sigma = -(-)^F \frac{1}{2} \text{Tr} \log \Delta, \quad (2.2)$$

where F denotes the fermion number of these fields (which is odd for fermions and even for bosons). Unfortunately the evaluation of the right-hand side of this expression is in general quite difficult, and explicit results are typically known only for background fields which are sufficiently simple.

Calculations are easier if one is only interested in those parts of Σ which are the most sensitive to very short-distance physics. In this case very general results can be obtained by using the Gilkey-DeWitt heat-kernel methods. For instance, the parts of Σ which depend the most strongly on the mass matrix, m , (in the limit that the eigenvalues of m are large) can be written as

$$\Sigma_{UV} = \frac{1}{2}(-)^F \left(\frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} \Gamma(k - n/2) \text{tr} [m^{n-2k} a_k] \quad (2.3)$$

where g is the determinant of the metric, $\Gamma(z)$ is Euler's gamma function and n is the number of spacetime dimensions. The a_k are local quantities constructed from k powers of the background curvature tensor, as well as of the other background fields. We stop the sum at $k = [n/2]$ — where $[n/2]$ denotes the largest integer which is $\leq n/2$ — since our interest is only in those terms which do not involve negative powers of m . Although all $[n/2]$ coefficients a_k are required to completely describe the UV properties of an n -dimensional theory, for practical reasons we calculate here only the first three (the number of terms in each a_k increases exponentially with n , c.f. eqs. (2.5) and (2.6)). Potential ultra-violet divergences in this expression are regulated by taking n to approach continuously the integer value of interest.

Very general explicit expressions for the first few a_k are known in some circumstances. Consider, for example, N real fields, ϕ^i , whose field equation when linearized about the background configuration is

$$\Delta_j^i \phi^j = (-\square + m^2 + X)_j^i \phi^j = 0, \quad (2.4)$$

where $\square = g^{MN} D_M D_N$ is constructed from background-covariant derivatives, D_M , and the quantity X_j^i is a local background-field dependent quantity. Using the heat kernel expansion, it is possible to show that the first few a_k , are given by:¹

$$\begin{aligned} a_0 &= I \\ a_1 &= -\frac{1}{6}(RI + 6X) \\ a_2 &= \frac{1}{360} (2R_{MNPQ}R^{MNPQ} - 2R_{MN}R^{MN} + 5R^2 - 12\square R) I \\ &\quad + \frac{1}{6}RX + \frac{1}{2}X^2 - \frac{1}{6}\square X + \frac{1}{12}Y_{MN}Y^{MN} \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} a_3 = \frac{1}{7!} & \left(-18\square^2 R + 17D_M R D^M R - 2D_L R_{MN} D^L R^{MN} - 4D_L R_{MN} D^N R^{ML} \right. \\ & + 9D_K R_{MNL} D^K R^{MNL} + 28R\square R - 8R_{MN}\square R^{MN} + 24R^M{}_N D^L D^N R_{ML} \\ & + 12R_{MNL} \square R^{MNL} - \frac{35}{9}R^3 + \frac{14}{3}R R_{MN} R^{MN} - \frac{14}{3}R R_{MNPQ} R^{MNPQ} \\ & + \frac{208}{9}R^M{}_N R_{ML} R^{NL} - \frac{64}{3}R^{MN} R^{KL} R_{MKNL} + \frac{16}{3}R^M{}_N R_{MKLP} R^{NKLP} \\ & \left. - \frac{44}{9}R^{AB}{}_MN R_{ABKL} R^{MNKL} - \frac{80}{9}R^A{}_B{}^M{}_N R_{AKMP} R^{BKNP} \right) I \\ & + \frac{1}{360} \left(8D_M Y_{NK} D^M Y^{NK} + 2D^M Y_{NM} D_K Y^{NK} + 12Y^{MN} \square Y_{MN} \right. \\ & - 12Y^M{}_N Y^N{}_K Y^K{}_M - 6R^{MNKL} Y_{MN} Y_{KL} + 4R^M{}_N Y_{MK} Y^{NK} \\ & - 5R Y^{MN} Y_{MN} - 6\square^2 X + 60X\square X + 30D_M X D^M X - 60X^3 \\ & - 30X Y^{MN} Y_{MN} + 10R\square X + 4R^{MN} D_M D_N X + 12D^M R D_M X - 30X^2 R \\ & \left. + 12X \square R - 5X R^2 + 2X R_{MN} R^{MN} - 2X R_{MNPQ} R^{MNPQ} \right), \end{aligned} \quad (2.6)$$

where I is the $N \times N$ identity matrix for the space of fields of interest, and Y_{MN} is the matrix-valued quantity defined by the expression $Y_{MN}{}^i{}_j \phi^j = [D_M, D_N]\phi^i$. Y_{MN}

¹Our metric is ‘mostly plus’ and we adopt Weinberg’s curvature conventions [13] (which differ from those of Misner Thorne and Wheeler [14] only in the overall sign of the curvature tensors).

may be expressed explicitly in terms of the Riemann tensor and any background gauge fields, A_M^a , as:

$$Y_{MN} = -iF_{MN}^a t_a - \frac{i}{2}R_{MN}^{AB}J_{AB}, \quad (2.7)$$

where t_a and J_{AB} are the field-appropriate matrices which generate gauge and Lorentz transformations, and F_{MN}^a is the background gauge field strength. In particular, for canonically-normalized gauge bosons, we take the gauge group generators to include a factor of the corresponding gauge coupling, g_a . Here we use indices A, B, \dots for the tangent frame, M, N, \dots for world indices and lower-case indices to label gauge-group generators.

Notice that there is an ambiguity in how the mass, m , enters into the above expressions, because the two quantities X and m^2 only enter through their sum: $X + m^2$. As a consequence there are two ways to use these formulae. On the one hand, one can lump the physical mass into X and regard the explicit m dependence of eq. (2.3) as being an infrared regulator which is taken to zero at the end of the calculation. In this case only the term with $k = n/2$ survives and the m dependence of Σ is completely contained within the X dependence of $a_{n/2}$. Alternatively one can exclude m^2 from X , in which case the large- m dependence of Σ is explicit in eq. (2.3).

We may use the equivalence of these two points of view to derive an identity which relates the Gilkey coefficients for X to those for $X + m^2$. The simplest way to do so is to compute the divergent part of eq. (2.3) using the result $\Gamma(-k - \epsilon) = (-)^k/(k!\epsilon) + \dots$, for ϵ an infinitesimal and k a non-negative integer. For odd n this leads to the old one-loop-finiteness result at one loop in dimensional regularization [15]. For even n , comparing the result for the coefficient of $1/\epsilon$ with and without including m^2 in X leads to the following identity:

$$\text{tr}[a_{n/2}(X + m^2)] = \sum_{k=0}^{n/2} \frac{(-)^{k-n/2}}{(n/2 - k)!} \text{tr}[m^{n-2k} a_k(X)]. \quad (2.8)$$

For instance, for $n = 4$ and $n = 6$ this reduces to

$$\begin{aligned} \text{tr}[a_2(X + m^2)] &= \text{tr}[a_2(X)] - \text{tr}[m^2 a_1(X)] + \frac{1}{2} \text{tr}[m^4 a_0(X)] \\ \text{tr}[a_3(X + m^2)] &= \text{tr}[a_3(X)] - \text{tr}[m^2 a_2(X)] + \frac{1}{2} \text{tr}[m^4 a_1(X)] - \frac{1}{6} \text{tr}[m^6 a_0(X)], \end{aligned} \quad (2.9)$$

which may be verified using the explicit expressions of eq. (2.5).

These formulae show that the coefficient of the leading power of m can be computed by evaluating the first few coefficients, a_k , *without* including m explicitly into the quantity X . Provided that the contributions of the would-be Goldstone bosons and ghosts all share the same m (as we show in detail below) we may obtain the results for massive fields by summing appropriate results for massless fields.

We now use this approach to evaluate the first few coefficients, $\text{tr}(a_k)$ ($k = 0, 1, 2$), in n spacetime dimensions for particles having spin zero, one-half, one, three-halves and two, as well as for the rank-two antisymmetric gauge potential which appears in supergravity models. Although our real interest is to applications with massive fields, we provide the results for massless fields which are required as intermediate steps in the calculation.

2.2 Spin 0

The lagrangian for a set of N_0 real scalar fields, denoted collectively by ϕ , is given by

$$\frac{1}{e} \mathcal{L}_0 = -\frac{1}{2} \phi (-\square + m^2 + \xi R) \phi \quad (2.10)$$

where in general both m^2 and ξ are arbitrary constant $N_0 \times N_0$ matrices, and as usual $e = \sqrt{-g}$. We here assume for simplicity that m^2 and ξ commute with one another, so a basis of fields exists for which both are diagonal. A case of particular interest is the massless, minimally-coupled case, $\xi = m^2 = 0$, such as would be enforced by a Goldstone-boson symmetry $\phi \rightarrow \phi + \text{constant}$. Alternatively, the case $m^2 = 0$ and

$$\xi = -\frac{(n-2)}{4(n-1)} I \quad (2.11)$$

describes a conformally-invariant coupling for all N_0 scalars.

For scalars we have $Y_{MN} = -iF_{MN}^a t_a$, where t_a is the gauge-group generator acting on the scalars of any background gauge group, under which the scalars are assumed to transform in a representation \mathcal{R}_0 . If this representation contains N_0 real scalars, then we have $\text{tr}(I) = N_0$. For $X = \xi R$ we find

$$\begin{aligned} \text{tr}_0(a_0) &= N_0 \\ \text{tr}_0(a_1) &= -\left(\text{tr} \xi + \frac{N_0}{6}\right) R \\ \text{tr}_0(a_2) &= \frac{N_0}{180} \left[R_{MNPQ} R^{MNPQ} - R_{MN} R^{MN} \right] + \frac{1}{2} \text{tr} \left[\left(\xi + \frac{1}{6} \right)^2 \right] R^2 \\ &\quad - \frac{1}{6} \text{tr} \left(\xi + \frac{1}{5} \right) \square R - \frac{g_a^2}{12} C(\mathcal{R}_0) F_{MN}^a F_a^{MN}. \end{aligned} \quad (2.12)$$

Here $\text{tr } \xi^k = N_0 \xi_0^k$ if all scalars share the same coupling to R (*i.e.* if $\xi = \xi_0 I$), and $\text{tr } [t_a t_b] = g_a^2 C(\mathcal{R}_0) \delta_{ab}$, where $C(\mathcal{R}_0)$ is the Dynkin index for the scalar representation \mathcal{R}_0 . (Our normalization is such that $C(F) = k/2$ or $C(A) = Nk$, respectively, for k fields in the fundamental or adjoint representations of $SU(N)$.)

2.3 Spin 1/2

We take the lagrangian for $N_{1/2}$ spin-half particles to be

$$\frac{1}{e} \mathcal{L}_{1/2} = -\frac{1}{2} \bar{\psi} (\not{D} + m) \psi, \quad (2.13)$$

where $\not{D} = \Gamma^M D_M$ with Γ^M denoting the $d \times d$ Dirac matrices in n dimensions. In n dimensions $d = 2^{[n/2]}$ where $[n/2]$ is the largest integer which is less than or equal to $n/2$. Since different kinds of spinors are possible in different spacetime dimensions, it proves useful to define a new quantity, $\tilde{d} = 2d/\zeta$, where the pre-factor of 2 comes because we count real fields, and $\zeta = 1, 2$, or 4 depending on whether the spinors in question are Dirac, Majorana or Weyl, or Majorana-Weyl.²

In order to put the operator Δ into a form for which eq. (2.5) applies, we use the fact that (assuming there are no gauge or Lorentz anomalies) $\log \det(\not{D} + m) = \frac{1}{2} \log \det(m^2 - \not{D}^2)$, which implies

$$\begin{aligned} i\Sigma_{1/2} &= \frac{1}{4} \text{Tr} \log (m^2 - \not{D}^2) \\ &= \frac{1}{4} \text{Tr} \log \left(-\square + m^2 - \frac{1}{4} R + \frac{i}{2} \Gamma^{AB} F_{AB}^a t_a \right), \end{aligned} \quad (2.14)$$

where we use the spin-half result $J_{AB} = -\frac{i}{2} \Gamma_{AB}$, with $\Gamma_{AB} = \frac{1}{2} [\Gamma_A, \Gamma_B]$. Thus, we see that eq. (2.5) may be applied if we use $X = -\frac{1}{4} R I + \frac{i}{2} \Gamma^{AB} F_{AB}^a t_a$, and divide the overall result by 2 (because of the extra factor of 1/2 in eq. (2.14) relative to eq. (2.2)). Here I denotes the $\mathcal{N}_{1/2} \times \mathcal{N}_{1/2}$ unit matrix, with $\mathcal{N}_{1/2} = N_{1/2} \tilde{d}$.

Using eq. (2.7), we find in this way

$$\text{tr } (Y_{MN} Y^{MN}) = -\tilde{d} g_a^2 C(\mathcal{R}_{1/2}) F_{MN}^a F_a^{MN} - \frac{1}{8} \mathcal{N}_{1/2} R_{MNPQ} R^{MNPQ}. \quad (2.15)$$

²For a discussion on the allowed spinors in spacetimes of arbitrary dimension and signature, see for example [16].

This leads to the following values for a_k :

$$\begin{aligned}
\text{tr}_{1/2}(a_0) &= \frac{\mathcal{N}_{1/2}}{2} \\
\text{tr}_{1/2}(a_1) &= \frac{\mathcal{N}_{1/2}}{24} R \\
\text{tr}_{1/2}(a_2) &= \frac{\mathcal{N}_{1/2}}{360} \left[-\frac{7}{8} R_{MNPQ} R^{MNPQ} - R_{MN} R^{MN} + \frac{5}{8} R^2 + \frac{3}{2} \square R \right] \\
&\quad + \frac{\tilde{d}g_a^2}{12} C(\mathcal{R}_{1/2}) F_{MN}^a F_a^{MN}.
\end{aligned} \tag{2.16}$$

2.4 Spin 1

For spins higher than 1/2 the massless and massive cases must be handled separately, due to the different number of spin states which are involved in these two cases. This is also related to the need for gauge symmetries for these higher spins [17], and the possibility of mixing between higher-spin and lower-spin fields (*i.e.* the Anderson-Higgs-Kibble mechanism). In order to be explicit we first present the massless case.

Massless Spin 1

We start by dividing the total gauge field into a background component, A_M^a , and a fluctuation, \mathcal{A}_M^a , according to $a_M^a = A_M^a + \mathcal{A}_M^a$. In terms of these fields the gauge field strength for the full field, a_M^a , becomes

$$f_{MN}^a = F_{MN}^a + D_M \mathcal{A}_N^a - D_N \mathcal{A}_M^a + c_{bc}^a \mathcal{A}_M^b \mathcal{A}_N^c, \tag{2.17}$$

where D_M is the background covariant derivative built from the background gauge connection, A_M^a , and Christoffel symbol, and as before F_{MN}^a is the background field-strength tensor. As usual, the fluctuation, \mathcal{A}_M^a , is chosen to transform in the adjoint representation under background gauge transformations — and so $(t_a)_{bc} = -ic_{abc}$ — as well as transforming as a vector under background coordinate transformations.

It is convenient to fix the spin-1 gauge invariance using a background-covariant gauge-averaging term,

$$\frac{1}{e} \mathcal{L}_V^{gf} = -\frac{1}{2\xi_1} (D^M \mathcal{A}_M^a)^2, \tag{2.18}$$

where D_M denotes the background-covariant derivative built from the background gauge field and Christoffel symbols. Then expanding the gauge-field lagrangian,

$$\frac{1}{e} (\mathcal{L}_V + \mathcal{L}_V^{gf}) = - \left[\frac{1}{4} f_{MN}^a f_a^{MN} + \frac{1}{2\xi_1} (D^M \mathcal{A}_M^a)^2 \right], \tag{2.19}$$

to second order in \mathcal{A}_M^a and choosing the background-covariant Feynman gauge ($\xi_1 = 1$), the part of the lagrangian which is quadratic in the fluctuations, $\mathcal{L}_{\mathcal{A}}$, becomes

$$\frac{1}{e} \mathcal{L}_{\mathcal{A}} = -\frac{1}{2} \mathcal{A}_a^M \left[-\square g_{MN} \delta^{ab} - Y_{MN}^{ab} + c_c{}^{ab} F_{MN}^c \right] \mathcal{A}_b^N, \quad (2.20)$$

where as before $[D_M, D_N] \mathcal{A}^{aN} = Y_{MN}^{ab} \mathcal{A}_b^N$.

For a vector field the Lorentz generators are $(J^{AB})_{CD} = -i(\delta_C^A \delta_D^B - \delta_D^A \delta_C^B)$, and so we see that $[D_M, D_N] \mathcal{A}_b^N = R_{MN} \mathcal{A}_b^N - i F_{MN}^a (t_a)_b{}^c \mathcal{A}_c^N$. The one-loop contribution due to vector loops is then given by:

$$\begin{aligned} i\Sigma_V &= -\frac{1}{2} \log \det \left[\Delta_{MN}^{a b} \right] \\ &= -\frac{1}{2} \log \det \left[-\square \delta_{MN}^{a b} - R_{MN}^{a b} + 2i F_{MN}^{cM} (t_c)_b{}^a \right]. \end{aligned} \quad (2.21)$$

We can now see that $X_{MN}^{a b} = -\eta R_{MN}^{a b} + 2i (t_c)_b{}^a F_{MN}^{cM}$, where $\eta = \pm 1$ is a useful constant to include for later purposes. For the case considered here we see that $\eta = 1$, whereas when we consider the ghosts associated with spin-2 particles we will find that $\eta = -1$.

For N_1 vector fields, we therefore find that $\text{tr}_V(X) = -\eta N_1 R$ and $\text{tr}_V(X^2) = N_1 R_{MN} R^{MN} + 4g_a^2 C(A) F_{MN}^a F_a^{MN}$, where $C(A)$ is the Dynkin index for N_1 fields in the adjoint representation. Similarly, $\text{tr}_V(Y_{MN} Y^{MN}) = -N_1 R_{MNPQ} R^{MNPQ} - ng_a^2 C(A) F_{MN}^a F_a^{MN}$ and $\text{tr}_V(I) = nN_1$. These imply the following results for vector fields in n spacetime dimensions:

$$\begin{aligned} \text{tr}_V(a_0) &= nN_1 \\ \text{tr}_V(a_1) &= \left(\eta - \frac{n}{6} \right) N_1 R \\ \text{tr}_V(a_2) &= \frac{N_1}{360} \left[(2n-30) R_{MNPQ} R^{MNPQ} + (180-2n) R_{MN} R^{MN} + (5n-60\eta) R^2 \right. \\ &\quad \left. + (60\eta-12n) \square R \right] + \frac{g_a^2}{12} (24-n) C(A) F_{MN}^a F_a^{MN}. \end{aligned} \quad (2.22)$$

Since we work in a covariant gauge, to this result must be added the contributions of the ghosts. For the gauge chosen, the gauge fixing condition $f^a = D^M \mathcal{A}_M^a$ varies under gauge transformations according to $\delta f^a = \square \epsilon^a$. Consequently, the lagrangian for the gauge ghosts is

$$\frac{1}{e} \mathcal{L}_{Vgh} = -\omega_a^* (-\square) \omega^a, \quad (2.23)$$

where the ω^a are complex fields obeying Fermi statistics. Since this has the same form as the spin zero lagrangian discussed above (specialized to $\xi = 0$), for the ghosts

we may simply adopt the spin-0 results for the a_k , with $N_0 \rightarrow N_1$ and multiplied by an overall factor of -2 .

Adding the results for vector fields ($\eta = +1$) and ghosts gives the contribution of physical spin-1 states. Thus, we obtain for massless spin-1 particles:

$$\begin{aligned} \text{tr}_1(a_0) &= N_1(n-2) \\ \text{tr}_1(a_1) &= \frac{N_1}{6}(8-n)R \\ \text{tr}_1(a_2) &= \frac{N_1}{180} \left[(n-17)R_{MNPQ}R^{MNPQ} + (92-n)R_{MN}R^{MN} \right] + \frac{N_1}{72}(n-14)R^2 \\ &\quad + \frac{N_1}{30}(7-n)\square R + \frac{g_a^2}{12}(26-n)C(A)F_{MN}^aF_a^{MN}. \end{aligned} \quad (2.24)$$

Massive Spin 1

If the gauge symmetry is spontaneously broken by the expectation of a scalar field, $\langle \phi^i \rangle = v^i$, then the previous discussion is complicated because the part of the lagrangian quadratic in fluctuations acquires cross terms between the vector and scalar fields of the form $\mathcal{A}_M^a t_a \partial^M \phi$. These terms reflect the physical process whereby the spin-1 particles acquire masses by absorbing the scalar fields through the Anderson-Higgs-Kibble mechanism.

In this case the same analysis as above can be performed provided we average over a more general gauge condition: $f^a = D^M \mathcal{A}_M^a + c v \cdot t^a \phi$, with the constant c chosen to remove the cross terms between \mathcal{A}_M^a and $\partial_M \phi$. This simply results in the addition of the same mass matrix μ^2 to the differential operator $\Delta = -\square + X$ for the vector fields and the ghost fields. This process also results in the would-be Goldstone bosons (*i.e.* the scalar fields which mixed with the gauge fields) acquiring the same mass matrix, μ^2 as also appears in the vector-field and ghost actions [18].

The upshot for massive spin-1 particles is therefore to add the result for N_1 massless spin-1 particles to that of N_1 massless scalar fields, with $\xi = 0$. This leads to the following contributions if the mass μ^2 , is not included in X :

$$\begin{aligned} \text{tr}_{1m}(a_0) &= N_1(n-1) \\ \text{tr}_{1m}(a_1) &= \frac{N_1}{6}(7-n)R \\ \text{tr}_{1m}(a_2) &= \frac{N_1}{180} \left[(n-16)R_{MNPQ}R^{MNPQ} + (91-n)R_{MN}R^{MN} \right] + \frac{N_1}{72}(n-13)R^2 \\ &\quad + \frac{N_1}{30}(6-n)\square R + \frac{g_a^2}{12}(25-n)C(A)F_{MN}^aF_a^{MN}. \end{aligned} \quad (2.25)$$

2.5 Antisymmetric Tensors

We next consider in detail the antisymmetric rank-2 gauge potential, B_{MN} , which appears in supergravity models. As before we first treat the massless case, and then move on to massive particles. We also quote the results for massless antisymmetric tensors of arbitrary rank, as taken from ref. [10].

Massless Antisymmetric Tensors

The appropriate lagrangian for this field is

$$\frac{1}{e} \mathcal{L}_B = -\frac{1}{12} H_{MNP} H^{MNP}, \quad (2.26)$$

where $H_{MNP} = D_{[M} B_{NP]} = 2(D_M B_{NP} + D_N B_{PM} + D_P B_{MN})$, and to this we add the gauge-fixing term $\frac{1}{e} \mathcal{L}_B^{gf} = -\frac{1}{2\xi_B} (D_M B^{MN})^2$. Choosing the gauge parameter to be $\xi_B = 1/4$, we obtain the lagrangian

$$\frac{1}{e} (\mathcal{L}_B + \mathcal{L}_B^{gf}) = -B_{MN} \left(-\square \bar{\delta}_{PQ}^{MN} + 2R_P^M \delta_Q^N - 2R_P^M \delta_Q^N \right) B^{PQ}. \quad (2.27)$$

Here, $\bar{\delta}_{PQ}^{MN} = \frac{1}{2}(\delta_P^M \delta_Q^N - \delta_Q^M \delta_P^N)$ is the appropriate identity matrix for a rank-2 antisymmetric tensor. The differential operator which possesses the correct symmetries for this field is thus seen to be

$$\Delta^{MN}_{PQ} = -\square \bar{\delta}_{PQ}^{MN} + (R_P^M \delta_Q^N - R_Q^N \delta_P^M) - \frac{1}{2}(R_P^M \delta_Q^N - R_P^N \delta_Q^M + R_Q^N \delta_P^M - R_Q^M \delta_P^N), \quad (2.28)$$

and so

$$X^{MN}_{PQ} = (R_P^M \delta_Q^N - R_Q^N \delta_P^M) - \frac{1}{2}(R_P^M \delta_Q^N - R_P^N \delta_Q^M + R_Q^N \delta_P^M - R_Q^M \delta_P^N). \quad (2.29)$$

Similarly Y_{MN} is given by

$$(Y_{MN})^{AB}_{CD} = \frac{1}{2}(R^A_{CMN} \delta_D^B \mp R^A_{DMN} \delta_C^B + R^B_{DMN} \delta_C^A \mp R^B_{CMN} \delta_D^A), \quad (2.30)$$

where for later convenience we also give here the result (bottom sign) for the rank-2 symmetric tensor field.

Using these expressions for X and Y_{MN} , and taking there to be N_a such antisymmetric gauge potentials, we obtain

$$\begin{aligned} \text{tr}_B(X) &= N_a(2-n)R \\ \text{tr}_B(X^2) &= N_a \left[R_{MNPQ} R^{MNPQ} + (n-6)R_{MN} R^{MN} + R^2 \right] \\ \text{tr}_B(Y_{MN} Y^{MN}) &= N_a(2-n)R_{MNPQ} R^{MNPQ}, \end{aligned} \quad (2.31)$$

and so are led to the following results for $\text{tr}(a_k)$:

$$\begin{aligned}
\text{tr}_B(a_0) &= \frac{N_a}{2}n(n-1) \\
\text{tr}_B(a_1) &= -\frac{N_a}{12}(n^2 - 13n + 24)R \\
\text{tr}_B(a_2) &= N_a \left[\frac{1}{360}(16-n)(15-n)R_{MNPQ}R^{MNPQ} \right. \\
&\quad - \frac{1}{360}(n^2 - 181n + 1080)R_{MN}R^{MN} + \frac{1}{144}(n^2 - 25n + 120)R^2 \\
&\quad \left. - \frac{1}{60}(n^2 - 11n + 20)\square R \right]. \tag{2.32}
\end{aligned}$$

To these expressions must be added the contributions of the ghosts. The anti-symmetric tensor gauge transformations are $\delta B_{MN} = D_M \Lambda_N - D_N \Lambda_M$, where Λ_M is itself only defined up to a gauge transformation: $\Lambda_M \rightarrow \Lambda_M + D_M \Phi$. We therefore average over the secondary gauge-fixing condition $f = D_M \Phi^M$, where D_M is the appropriate background-covariant derivative. Introducing ghosts and ghost-for-ghosts for these symmetries, we acquire the ghost counting of ref. [19], which states that each initial tensor gauge potential gives rise to a complex, fermionic vector ghost, ω^M , and three real, scalar, bosonic ghosts-for-ghosts,³ ϕ^i . Their lagrangians are given by

$$\begin{aligned}
\frac{1}{e} \mathcal{L}_{BVgh} &= -\omega_M^*(-\square \delta_N^M - R_N^M)\omega^N, \\
\frac{1}{e} \mathcal{L}_{BSgh} &= -\frac{1}{2}\phi_i(-\square)\phi^i. \tag{2.33}
\end{aligned}$$

The contributions of the vector ghosts to a_k is therefore obtained by replacing $N_1 \rightarrow -2N_a$ in the result given above for vector fields (with $\eta = +1$). Similarly, the scalar ghosts are obtained from the spin-0 result quoted above, with the replacements $N_0 \rightarrow 3N_a$ and $\xi \rightarrow 0$.

Summing the contribution of the rank-2 tensor and its ghosts leads to the following expression for the physical massless particles associated with these antisymmetric

³The reason we do not obtain four scalar ghosts, as a naive ghost counting would imply, has to do with the fact the gauge-fixing function $G_N = D^M B_{MN}$ satisfies the constraint $D^N G_N = 0$. A more detailed discussion of this point can be found in [19].

tensor fields:

$$\begin{aligned}
\text{tr}_a(a_0) &= \frac{N_a}{2}(n-2)(n-3) \\
\text{tr}_a(a_1) &= -\frac{N_a}{12}(n^2 - 17n + 54)R \\
\text{tr}_a(a_2) &= \frac{N_a}{360} \left[(n^2 - 35n + 306)R_{MNPQ}R^{MNPQ} - (n^2 - 185n + 1446)R_{MN}R^{MN} \right] \\
&\quad + \frac{N_a}{144}(n^2 - 29n + 174)R^2 - \frac{N_a}{60}(n^2 - 15n + 46)\square R.
\end{aligned} \tag{2.34}$$

Massive Particles

The particles associated with antisymmetric tensor fields can also acquire mass through an Anderson-Higgs-Kibble mechanism, in which the antisymmetric tensor particle ‘eats’ an ordinary gauge field, V_M [20]. As before, a modification of the gauge choice is required in this case in order not to have mixing terms of the form $B^{MN}\partial_M V_N$. As we now show, the contribution of each massive tensor particle is given by adding the above result for a massless particle to the result for an $\eta = +1$ massless abelian — so with $C(A) = 0$ — gauge field (including its ghosts).

To demonstrate this explicitly, we start with the lagrangian

$$\frac{1}{e} \mathcal{L}_{mB} = -\frac{1}{12}H_{MNP}H^{MNP} - \frac{1}{4}(V_{MN} - 2m B_{MN})^2, \tag{2.35}$$

where $V_{MN} = D_M V_N - D_N V_M$ is the field strength of the abelian gauge field V_M , and m is a constant with dimensions of mass. This lagrangian is invariant under

$$\begin{aligned}
\delta B_{MN} &= D_M \Lambda_N - D_N \Lambda_M \\
\delta V_M &= 2m \Lambda_M + 2 \partial_M \sigma,
\end{aligned} \tag{2.36}$$

where Λ_M and σ are arbitrary gauge parameters. As in the massless case, this set of gauge transformations is itself invariant under a gauge transformation,

$$\begin{aligned}
\delta \Lambda_M &= \partial_M \epsilon \\
\delta \sigma &= -m \epsilon,
\end{aligned} \tag{2.37}$$

for an arbitrary function ϵ . As before, therefore, we find that the ghosts themselves have ghosts. Note that in the limit $m \rightarrow 0$ the lagrangian decouples into the lagrangian for a massless antisymmetric tensor and a massless vector.

To fix the two gauge freedoms in eq. (2.36), and to remove unwanted mixing terms, we add to the lagrangian the gauge-fixing term

$$\frac{1}{e} \mathcal{L}_{mB}^{gf} = -2 \left(D^M B_{MN} - \frac{m}{2} V_N \right)^2 - \frac{1}{2} (D^M V_M)^2. \tag{2.38}$$

After adding this term to eq. (2.35) we find

$$\frac{1}{e}(\mathcal{L} + \mathcal{L}_{mB}^{gf}) = -B_{MN}(\Delta^{MN}{}_{PQ} + m^2 \bar{\delta}_{PQ}^{MN})B^{PQ} - \frac{1}{2}V_M(\Delta_N^M + m^2 \delta_N^M)V^N, \quad (2.39)$$

where Δ is the differential operator appropriate for the field it operates on; specifically, $\Delta_N^M = -\square \delta_N^M - R_N^M$, and $\Delta^{MN}{}_{PQ}$ is given by eq. (2.28).

The lagrangian for the ghosts is obtained by varying the gauge-fixing conditions appearing in eq. (2.38), and we thus find

$$\begin{aligned} \mathcal{L}_{mBgh} = & -\xi_N^*(-\square \delta_M^N + D_M D^N + m^2 \delta_M^N) \xi^M - \omega^*(-\square) \omega \\ & - m \xi_M^* D^M \omega + m \omega^* D_M \xi^M. \end{aligned} \quad (2.40)$$

Here, ξ_M and ω are the ghost fields associated with Λ_M and σ , respectively. To fix the gauge freedom implied by eq. (2.37), we add to the ghost lagrangian the term

$$\mathcal{L}_{mBgh}^{gf} = -(D_M \xi^M + m\omega)^* (D_N \xi^N + m\omega) \quad (2.41)$$

and so we find

$$\mathcal{L}_{mBgh} + \mathcal{L}_{mBgh}^{gf} = -\xi_N^* [(-\square + m^2) \delta_M^N + R_M^N] \xi^M - \omega^* (-\square + m^2) \omega. \quad (2.42)$$

Notice that the complex scalar ghost, ω , combines with the vector, V_N , to form the field content of a physical massless spin-1 particle.

The ghosts-for-ghosts lagrangian is similarly obtained, and as in the massless case we find three bosonic scalar ghosts-for-ghosts, with lagrangian

$$\mathcal{L}_{mBSgh} = -\frac{1}{2} \phi_i (-\square + m^2) \phi^i. \quad (2.43)$$

Except for the presence of mass terms, the lagrangian for a massive antisymmetric tensor is therefore the sum of a massless spin-1 lagrangian and a massless antisymmetric tensor lagrangian (including their ghosts). Thus, in calculating the a_k for a massive antisymmetric tensor, we simply need to add to the massless result given in the previous section the result for a massless spin-1 field. It is important to emphasize that such a sum — where we factor all mass terms out of X , as described in the § 2.1 — makes sense only because in the gauge we have chosen all particles share the same mass.

The result of this sum, for massive rank-2 tensor fields in n spacetime dimensions, is

$$\begin{aligned}\text{tr}_{am}(a_0) &= \frac{N_a}{2}(n-2)(n-1) \\ \text{tr}_{am}(a_1) &= -\frac{N_a}{12}(n^2 - 15n + 38)R \\ \text{tr}_{am}(a_2) &= \frac{N_a}{360} \left[(n^2 - 33n + 272)R_{MNPQ}R^{MNPQ} - (n^2 - 183n + 1262)R_{MN}R^{MN} \right] \\ &\quad + \frac{N_a}{144}(n^2 - 27n + 146)R^2 - \frac{N_a}{60}(n^2 - 13n + 32)\square R.\end{aligned}\quad (2.44)$$

Higher-Rank Antisymmetric Tensors

The result for a higher-rank massless skew-tensor gauge potential in n dimensions has been worked out in a similar fashion to the above [10]. This leads to the following results for the first few Gilkey coefficients for a massless 3-form gauge field (for $n > 4$ dimensions), specialized to Ricci-flat background geometries ($R_{MN} = 0$):

$$\begin{aligned}\text{tr}_{3a}(a_0) &= \frac{N_{3a}}{3!}(n-2)(n-3)(n-4) \\ \text{tr}_{3a}(a_1) &= 0 \\ \text{tr}_{3a}(a_2) &= \frac{N_{3a}}{1080}(n^3 - 54n^2 + 971n - 4164)R_{MNPQ}R^{MNPQ}.\end{aligned}\quad (2.45)$$

The analogous results for a massless 4-form gauge field (in $n > 5$ Ricci-flat dimensions) are given by:

$$\begin{aligned}\text{tr}_{4a}(a_0) &= \frac{N_{4a}}{4!}(n-2)(n-3)(n-4)(n-5) \\ \text{tr}_{4a}(a_1) &= 0 \\ \text{tr}_{4a}(a_2) &= \frac{N_{4a}}{4320}(n^4 - 74n^3 + 2051n^2 - 18634n + 52680)R_{MNPQ}R^{MNPQ}.\end{aligned}\quad (2.46)$$

These results for massive 1- and 2-forms suggest a short-cut for extending our results to the case of a massive p -form for arbitrary p , since they show that the Gilkey coefficients for a massive p -form are obtained by summing the contributions of a massless $(p-1)$ -form to that of a massless p -form. It can also be readily seen that the Gilkey coefficients for a massive spin-1 field are obtained by the replacement $n \rightarrow (n+1)$ in the massless formulae, and similarly for the antisymmetric 2-form. One way to see why this should give the correct result is to reason as follows. It is clear that (for a Minkowski-space background) a massless p -form in $(n+1)$ dimensions

and a massive p -form in n dimensions share the same little group, $SO(n-1)$, and transform in the same representation of this group. This connection can also be made more explicit by dimensionally reducing an $(n+1)$ -dimensional massless p -form on S^1 to obtain a Kaluza-Klein tower of massive p -forms in the lower-dimensional theory. Each massive field is thereby seen to contain the spin content of an n -dimensional massless p - and $(p-1)$ -form. A final check on this reasoning can be had using the results of ref. [10], which show that the first few Gilkey coefficients for a massless $(n+1)$ -dimensional p -form — and hence a massive n -dimensional p -form — are the same as the sum of the coefficients for a massless p - and $(p-1)$ -form in n dimensions.

2.6 Spin 3/2

Before proceeding with spin-3/2 and spin-2 particles, we first pause to establish a few of our supergravity conventions. Our starting point is the coupled Einstein/Rarita-Schwinger system. We take the spin-2 field to be described by the standard Einstein-Hilbert action, which in our conventions is

$$\frac{1}{e} \mathcal{L}_{EH} = -\frac{1}{2\kappa^2} R, \quad (2.47)$$

with $\kappa^2 = 8\pi G_N$. For the moment, we do not include a cosmological term; the generalization of the massless and massive spin-3/2 particle to the case of a nonzero cosmological constant is given in the appendix.

The spin-3/2 particle is described by a vector-spinor field, ψ_M , with a kinetic term given by the lagrangian

$$\frac{1}{e} \mathcal{L}_{VS} = -\frac{1}{2} \bar{\psi}_M \Gamma^{MNP} D_N \psi_P. \quad (2.48)$$

As before, we use indices A, B, \dots for the tangent frame, M, N, \dots for world indices and lower-case indices to label gauge-group generators. Conversion between tangent and world indices is accomplished using the vielbein, $e_M{}^A$. Here, $\Gamma^{ABC} = \frac{1}{6}[\Gamma^A \Gamma^B \Gamma^C + \dots]$ and $\Gamma^{AB} = \frac{1}{2}[\Gamma^A, \Gamma^B]$ are normalized completely antisymmetric combinations of gamma matrices.

The covariant derivative appearing in eq. (2.48) can involve background gauge fields in addition to the Christoffel connection, but only if the corresponding gauge symmetry does not commute with supersymmetry. Such transformations are particularly rich when there is more than one supersymmetry in the problem. Gravitini cannot carry charges for internal symmetries which commute with supersymmetry,

because for these the gravitino must share the charge of the graviton, which is neutral under all gauge transformations.

When there are no gauge fields in $D_M\psi_N$, it is straightforward to verify that the combination $\mathcal{L}_{VS} + \mathcal{L}_{EH}$ is invariant under the linearized supersymmetry transformations

$$\delta e_M^A = -\frac{\kappa}{4} \bar{\psi}_M \Gamma^A \epsilon + \text{c.c.}, \quad \delta \psi_M = \frac{1}{\kappa} D_M \epsilon. \quad (2.49)$$

When background gauge fields *are* present in $D_M\psi_N$, the combination $\mathcal{L}_{VS} + \mathcal{L}_{EH}$ varies into terms involving these gauge fields. These terms then cancel against variations of the gauge-field kinetic terms and with gauge-field-dependent terms in the gravitino transformation law. This shows that gauge fields for symmetries which do not commute with supersymmetry are special in that they are intimately related to the gravitini by supersymmetry.

Massless Spin 3/2

In order to put the spin-3/2 lagrangian into a form for which the general expressions for the Gilkey coefficients apply, it is convenient to use the following gauge-averaging term,

$$\frac{1}{e} L_{VS}^{gf} = -\frac{1}{2\xi_{3/2}} (\overline{\Gamma \cdot \psi}) \not{D}(\Gamma \cdot \psi). \quad (2.50)$$

With this term, and after making the field redefinition $\psi_M \rightarrow \psi_M + A\Gamma_M \Gamma \cdot \psi$, we find that the lagrangian simplifies in the desired way when we make the following choices for A and $\xi_{3/2}$:

$$A = \frac{1}{2-n} \quad \text{and} \quad \frac{1}{\xi_{3/2}} = \frac{2-n}{4}. \quad (2.51)$$

These choices allow the vector-spinor lagrangian to be written as

$$\frac{1}{e} (\mathcal{L}_{VS} + \mathcal{L}_{VS}^{gf}) = -\frac{1}{2} \bar{\psi}_M \not{D} \psi^M, \quad (2.52)$$

and so give the one-loop contribution

$$i\Sigma = \frac{1}{2} \log \det \left[(\not{D})^A_B \right] = \frac{1}{4} \log \det \left[(-\not{D}^2)^A_B \right]. \quad (2.53)$$

For a vector-spinor the Lorentz generators are

$$(J_{AB})^C_D = -\frac{i}{2} \Gamma_{AB} \delta_D^C - iI(\delta_A^C \eta_{BD} - \delta_B^C \eta_{AD}), \quad (2.54)$$

where I is the $\mathcal{N}_{3/2} \times \mathcal{N}_{3/2}$ identity matrix, corresponding to the $\mathcal{N}_{3/2} = N_{3/2} \tilde{d}$ (unwritten) non-vector components of ψ_M . (Recall $\tilde{d} = 2^{[n/2]+1}/\zeta$, where $\zeta = 1, 2$ and

4 for Dirac, Weyl (or Majorana) and Majorana-Weyl fermions.) Using the identity $\mathcal{D}^2 = \square + \frac{1}{4}[\Gamma^M, \Gamma^N][D_M, D_N]$, we find

$$[\mathcal{D}^2]_B^A = \left(\square + \frac{1}{4}R - \frac{i}{2}F_{CD}^a \Gamma^{CD} t_a \right) \delta_B^A - \frac{1}{2}R^A_{BCD} \Gamma^{CD}. \quad (2.55)$$

For simplicity of notation, we have suppressed writing the various identity matrices that appear in the above expression. From this we may read off the expression for X , given by

$$X_B^A = \left(-\frac{1}{4}R + \frac{i}{2}F_{CD}^a \Gamma^{CD} t_a \right) \delta_B^A + \frac{1}{2}R^A_{BMN} \Gamma^{MN}. \quad (2.56)$$

Taking appropriate traces, we obtain the results

$$\begin{aligned} \text{tr}_{VS}(X) &= -\frac{n}{4}\mathcal{N}_{3/2}R \\ \text{tr}_{VS}(X^2) &= \mathcal{N}_{3/2} \left[\frac{n}{16}R^2 + \frac{1}{2}R_{MNPQ}R^{MNPQ} \right] + \frac{n\tilde{d}g_a^2}{2}C(\mathcal{R}_{3/2})F_{MN}^a F_a^{MN} \\ \text{tr}_{VS}(Y_{MN}Y^{MN}) &= -\mathcal{N}_{3/2} \left(1 + \frac{n}{8} \right) R_{MNPQ}R^{MNPQ} - n\tilde{d}g_a^2 C(\mathcal{R}_{3/2})F_{MN}^a F_a^{MN}. \end{aligned} \quad (2.57)$$

$\mathcal{R}_{3/2}$ denotes, as usual, the Dynkin index for the representation of the gauge group carried by the spin-3/2 fields.

Combining these results, and remembering to multiply (as for the spin-1/2 case) eq. (2.5) by an overall factor of 1/2, we find

$$\begin{aligned} \text{tr}_{VS}(a_0) &= \frac{n}{2}\mathcal{N}_{3/2} \\ \text{tr}_{VS}(a_1) &= \frac{n}{24}\mathcal{N}_{3/2}R \\ \text{tr}_{VS}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7n}{8} \right) R_{MNPQ}R^{MNPQ} - nR_{MN}R^{MN} + \frac{5n}{8}R^2 + \frac{3n}{2}\square R \right] \\ &\quad + \frac{n\tilde{d}g_a^2}{12}C(\mathcal{R}_{3/2})F_{MN}^a F_a^{MN}. \end{aligned} \quad (2.58)$$

We next consider the contribution from the ghost fields. From the supersymmetry transformation rules, we see that $\delta(\Gamma \cdot \psi) = \frac{1}{\kappa}\mathcal{D}\epsilon$ and so there are two bosonic, Faddeev-Popov spinor ghosts with the lagrangian

$$\frac{1}{e}\mathcal{L}_{LVFPgh} = -\bar{\omega}^i \mathcal{D}\omega_i, \quad (2.59)$$

where $i = 1, 2$ labels the two ghosts. Since this has the same form as the spin-1/2 lagrangian used earlier, eq. (2.13), the Faddeev-Popov ghost result for $\text{tr}[a_k]$ is obtained by multiplying the massless spin-1/2 result by -2 .

In addition to the Faddeev-Popov ghosts, there is also a bosonic, Nielsen-Kallosh ghost [21] coming from the use of the operator \not{D} in the gauge-fixing lagrangian, eq. (2.50). The Nielsen-Kallosh ghost lagrangian is given by

$$\frac{1}{e} \mathcal{L}_{LVNKgh} = -\bar{\eta} \not{D} \eta. \quad (2.60)$$

This ghost therefore has a contribution to $\text{tr}[a_k]$ given by -1 times the massless spin-1/2 result.

Adding the results for the Faddeev-Popov and Nielsen-Kallosh ghosts to that of the vector-spinor, we obtain the following results for the contribution to $\text{tr}[a_k]$ by physical massless spin-3/2 states:

$$\begin{aligned} \text{tr}_{3/2}(a_0) &= \frac{\mathcal{N}_{3/2}}{2}(n-3) \\ \text{tr}_{3/2}(a_1) &= \frac{\mathcal{N}_{3/2}}{24}(n-3)R \\ \text{tr}_{3/2}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7}{8}(n-3) \right) R_{MNPQ}R^{MNPQ} - (n-3)R_{MN}R^{MN} \right. \\ &\quad \left. + \frac{5}{8}(n-3)R^2 + \frac{3}{2}(n-3)\square R \right] + \frac{\tilde{d}g_a^2}{12}(n-3)C(\mathcal{R}_{3/2})F_{MN}^aF_a^{MN}. \end{aligned} \quad (2.61)$$

Massive Spin 3/2

A spin-3/2 state acquires a mass through the existence of an off-diagonal coupling of the form $\bar{\chi}\Gamma\cdot\psi$ with a spin-1/2 Goldstone fermion state, χ . Choosing a gauge for which this term vanishes causes the super-Higgs mechanism to occur, through which the spin-3/2 particle ‘eats’ the fermion χ . Although χ vanishes in a unitary gauge, it remains in the theory in a covariant gauge much as does the would-be Goldstone boson for the massive spin-1 case.

To show explicitly how this process occurs, we assume that the part of the fermionic lagrangian which is quadratic in the fluctuations has the general form⁴

$$\begin{aligned} \frac{1}{e} \mathcal{L}_{mVS} &= -\bar{\psi}_M \Gamma^{MNP} D_N \psi_P - \bar{\chi} \not{D} \chi - \left[\bar{\psi} \cdot \Gamma(a \not{D} + b) \chi + \text{c.c.} \right] \\ &\quad - (c \bar{\psi}_M D^M \chi + \text{c.c.}) - m_{1/2} \bar{\chi} \chi - \mu_{3/2} \bar{\psi}_M \psi^M \\ &\quad + m_{3/2} \bar{\psi}_M \Gamma^{MN} \psi_N, \end{aligned} \quad (2.62)$$

⁴We follow here the approach of ref. [22] to identify the form of these couplings to quadratic order in a model-independent way.

where the parameters $a, b, c, m_{1/2}, m_{3/2}$, and $\mu_{3/2}$ are constrained by demanding that the action be invariant under linearized supersymmetry transformations. For simplicity we assume these parameters to be real, although in general some or all of these parameters may be complex, depending on whether the fermions are Majorana or Weyl in the supergravity of interest. Requiring invariance under the supersymmetry transformations

$$\delta\psi_M = \frac{1}{\kappa} D_M \epsilon + \mu \Gamma_M \epsilon \quad \text{and} \quad \delta\chi = f \epsilon, \quad (2.63)$$

then imposes the following constraints on the various parameters:

$$\begin{aligned} a = c = \mu_{3/2} &= 0 & b = \kappa f & \quad f^2 = (n-1)(n-2)\mu^2 \\ m_{1/2} &= n\kappa\mu & m_{3/2} &= (n-2)\kappa\mu. \end{aligned} \quad (2.64)$$

This leaves one free parameter — which we can take to be μ , f , or b — having the physical interpretation of being the supersymmetry breaking scale.

With these choices, the variation of the gravitino/goldstino lagrangian is

$$\frac{1}{e} \delta\mathcal{L}_{mVS} = \frac{1}{2\kappa} G^{MN} \bar{\psi}_M \Gamma_N \epsilon + \text{c.c.}, \quad (2.65)$$

where $G^{MN} = R^{MN} - \frac{1}{2} R g^{MN}$ is the Einstein tensor. This term is cancelled in the usual way by the variation of the Einstein-Hilbert action under the graviton transformation

$$\delta e_M^A = -\frac{\kappa}{4} \bar{\psi}_M \Gamma^A \epsilon + \text{c.c..} \quad (2.66)$$

To this lagrangian we add the gauge-fixing term

$$\frac{1}{e} \mathcal{L}_{mVS}^{gf} = -\bar{F}(\not{D} + \gamma)F, \quad (2.67)$$

where

$$F = \alpha \Gamma \cdot \psi + \beta \chi. \quad (2.68)$$

The constants α , β , and γ are chosen to ensure that the gauge-fixed lagrangian has the form

$$\frac{1}{e} (\mathcal{L}_{mVS} + \mathcal{L}_{mVS}^{gf}) = -\bar{\psi}'_M (\not{D} + m'_{3/2}) \psi'^M - \bar{\chi}' (\not{D} + m'_{1/2}) \chi' \quad (2.69)$$

where ψ'_M and χ' are given by

$$\chi' = A\chi + B\Gamma \cdot \psi \quad \text{and} \quad \psi'_M = \psi_M + C\Gamma_M \Gamma \cdot \psi + D\Gamma_M \chi, \quad (2.70)$$

where we again take the parameters A , B , C , and D to be real for simplicity. Note that the transformation of ψ_M is nonsingular provided $C \neq -1/n$. Using eq. (2.70) to evaluate the right-hand side of eq. (2.69) while using eqs. (2.62), (2.64), and (2.67) to evaluate the left-hand side, leads to the conditions

$$\begin{aligned} A &= \left(\frac{n-1}{n-2}\right)^{1/2}, & B = C &= -\frac{1}{2}, & D &= 0, & m'_{3/2} &= m'_{1/2} = (n-2)\kappa\mu, \\ \alpha &= -\frac{1}{2}\sqrt{n-1}, & \beta &= \frac{1}{\sqrt{n-2}}, & \gamma &= -(n-2)\kappa\mu. \end{aligned} \quad (2.71)$$

The ghost action consists of a Nielsen-Kallosh ghost, with lagrangian

$$\frac{1}{e} \mathcal{L}_{mVSNK} = -\bar{\omega}(\not{D} + \gamma)\omega, \quad (2.72)$$

as well as two Faddeev-Popov ghosts, with lagrangian

$$\frac{1}{e} \mathcal{L}_{mVSFP} = -\bar{\xi}_i \left[\not{D} + (n-2)\kappa\mu \right] \xi^i, \quad (2.73)$$

where $i = 1, 2$ labels the two ghosts. Dropping the primes, and defining $m = (n-2)\kappa\mu$, the complete lagrangian, eqs. (2.69), (2.72) and (2.73), becomes

$$\frac{1}{e} \mathcal{L}_{m3/2} = -\bar{\psi}_M(\not{D} + m)\psi^M - \bar{\chi}(\not{D} + m)\chi - \bar{\omega}(\not{D} - m)\omega - \bar{\xi}_i(\not{D} + m)\xi^i. \quad (2.74)$$

Since the heat-kernel coefficients are even under $m \rightarrow -m$, we see from this that a_k for a massive gravitino are given by the sum of the corresponding coefficients for a massless gravitino (including ghosts) plus those of a massless fermion. Summing the massive spin-1/2 result, eq. (2.16), with the spin-3/2 result, eq. (2.61), we obtain the following Gilkey coefficients for a massive spin-3/2 particle

$$\begin{aligned} \text{tr}_{m3/2}(a_0) &= \frac{\mathcal{N}_{3/2}}{2}(n-2) \\ \text{tr}_{m3/2}(a_1) &= \frac{\mathcal{N}_{3/2}}{24}(n-2)R \\ \text{tr}_{m3/2}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7}{8}(n-2) \right) R_{MNPQ}R^{MNPQ} - (n-2)R_{MN}R^{MN} \right. \\ &\quad \left. + \frac{5}{8}(n-2)R^2 + \frac{3}{2}(n-2)\square R \right] + \frac{g_a^2}{12}(n-2)\tilde{d}C(R_{3/2})F_{MN}^aF_a^{MN}. \end{aligned} \quad (2.75)$$

2.7 Spin 2

Finally, we turn to spin-2 particles. In order to maximize the utility of this section, we do so for the case where the lagrangian includes a cosmological constant, as is

typically true for non-supersymmetric theories (and for supersymmetric theories in four dimensions), and so start with the following action

$$\frac{1}{e} \mathcal{L}_{EH} = -\frac{1}{2\kappa^2} (R - 2\Lambda). \quad (2.76)$$

For situations where Λ represents the value of a scalar potential, V , evaluated at the classical background, we see from the above that $\Lambda = -\kappa^2 V$.

Although it is usually true that only a single spin-2 particle is massless in any given model, we include a parameter N_2 which counts the massive spin-2 states. We do so because there is typically more than one massive spin-2 state in the models of interest, typically arising as part of a Kaluza-Klein tower or as excited string modes.

Massless Spin 2

The lagrangian for a massless rank-two symmetric field is the Einstein-Hilbert action, eq. (2.76). As usual we write the metric as $g_{MN} + 2\kappa h_{MN}$, where g_{MN} is the background metric and h_{MN} are the fluctuations. Expanding to quadratic order in these fluctuations, and adding the gauge-fixing term

$$\frac{1}{e} \mathcal{L}_{EH}^{gf} = - \left(D^M h_{MN} - \frac{1}{2} D_N h^M_M \right)^2, \quad (2.77)$$

we obtain the standard result [9]

$$\begin{aligned} \frac{1}{e} (\mathcal{L}_{EH} + \mathcal{L}_{EH}^{gf}) = & \frac{1}{2} h^{MN} \left[\square h_{MN} + (R - 2\Lambda) h_{MN} - (h_{MA} R_N^A + h_{NA} R_M^A) \right. \\ & \left. - 2R_{MANB} h^{AB} \right] + h^{MN} R_{MN} h - \frac{1}{4} h \left[\square h + (R - 2\Lambda) h \right], \end{aligned} \quad (2.78)$$

where $h = g^{MN} h_{MN}$.

It is useful to decouple the scalar, h , from the traceless symmetric tensor $\phi_{MN} = h_{MN} - \frac{1}{n} h g_{MN}$, in this expression. In terms of these variables the lagrangian is

$$\begin{aligned} \frac{1}{e} (\mathcal{L}_{EH} + \mathcal{L}_{EH}^{gf}) = & \frac{1}{2} \phi^{MN} \left[\square \phi_{MN} + (R - 2\Lambda) \phi_{MN} - (\phi_{MA} R_N^A + \phi_{NA} R_M^A) \right. \\ & \left. - 2R_{MANB} \phi^{AB} \right] + \left(\frac{n-4}{n} \right) \phi^{MN} R_{MN} h \\ & - \left(\frac{n-2}{4n} \right) \left[h \square h + \left(\frac{n-4}{n} \right) R h^2 - 2\Lambda h^2 \right], \end{aligned} \quad (2.79)$$

which shows that these fields decouple if we make the assumption that the background metric is an Einstein space: $R_{MN} = \frac{1}{n} R g_{MN}$. Although it seems restrictive,

the assumption that the background be an Einstein space is actually reasonably general due to the observation that we lose no generality if we simplify the one-loop action by using the classical equations of motion. We are always free to do so because it is always possible to use a field redefinition to remove any term in the one-loop action which vanishes when the classical equations are used [2].⁵ In the presence of a scalar potential, V , (or cosmological constant, $\Lambda = -\kappa^2 V$), the classical equations may often be written $G_{MN} + \Lambda g_{MN} = 0$, or $R_{MN} = [2\Lambda/(n-2)]g_{MN}$, and for any such a configuration our analysis applies.

With this assumption, and canonically normalizing the scalar mode by taking $\phi = [(n-2)/(2n)]^{1/2}h$, we arrive at the desired expression:

$$\begin{aligned} \frac{1}{e}(\mathcal{L}_{EH} + \mathcal{L}_{EH}^{gf}) = & -\frac{1}{2}\phi^{MN}\left[-\square\bar{\delta}_{MN}^{AB} + 2R_M{}^A{}_N{}^B + (R_M^A\delta_N^B + R_N^A\delta_M^B)\right. \\ & \left.-(R-2\Lambda)\bar{\delta}_{MN}^{AB}\right]\phi_{AB} - \frac{1}{2}\phi\left[\square + \left(\frac{n-4}{n}\right)R - 2\Lambda\right]\phi, \end{aligned} \quad (2.80)$$

where $\bar{\delta}_{AB}^{MN} = \frac{1}{2}(\delta_A^M\delta_B^N + \delta_B^M\delta_A^N) - \frac{1}{n}g^{MN}g_{AB}$ is the unit matrix appropriate for a traceless symmetric tensor. Notice the presence of the well-known ‘wrong’ sign for the kinetic term of the scalar mode ϕ .

We may now separately compute the contributions of ϕ and ϕ_{MN} to the heat-kernel coefficients, a_k . From eq. (2.80), the symmetric traceless differential operator appropriate for ϕ_{MN} is seen to be

$$\begin{aligned} \Delta^{MN}{}_{PQ} = & -\left[\square + (R-2\Lambda)\right]\bar{\delta}_{PQ}^{MN} + (R_P^M{}^N{}_Q + R_Q^N{}^M{}_P) - \frac{4}{n}(g_{PQ}R^{MN} + g^{MN}R_{PQ}) \\ & + \frac{1}{2}(R_P^M\delta_Q^N + R_P^N\delta_Q^M + R_Q^N\delta_P^M + R_Q^M\delta_P^N) + \frac{4}{n^2}g^{MN}g_{PQ}R, \end{aligned} \quad (2.81)$$

from which the expression for X can be read off directly. Taking traces of the relevant

⁵Although it is always possible to simplify (without loss of generality) the one-loop action using the classical equations – see below – by excluding things like scalar gradients or background F_{MN}^a this assumption restricts the kinds of solutions to the classical equations we may entertain.

quantities, we find

$$\begin{aligned}
\text{tr}_{\text{symtr}}(X) &= N_2 \left[-\frac{1}{2n}(n+2)(n^2-3n+4)R + (n+2)(n-1)\Lambda \right] \\
\text{tr}_{\text{symtr}}(X^2) &= N_2 \left[3R_{MNPQ}R^{MNPQ} + \frac{1}{n}(n^2-2n-32)R_{MN}R^{MN} \right. \\
&\quad \left. + \frac{1}{2n^2}(n^4-3n^3+16n+32)R^2 \right. \\
&\quad \left. - \frac{2}{n}(n+2)(n^2-3n+4)\Lambda R + 2(n+2)(n-1)\Lambda^2 \right] \\
\text{tr}_{\text{symtr}}(Y_{MN}Y^{MN}) &= -N_2(n+2)R_{MNPQ}R^{MNPQ}. \tag{2.82}
\end{aligned}$$

Applying eq. (2.5), we arrive at the following expressions for $\text{tr}[a_k]$:

$$\begin{aligned}
\text{tr}_{\text{symtr}}(a_0) &= \frac{N_2}{2}(n+2)(n-1) \\
\text{tr}_{\text{symtr}}(a_1) &= N_2 \left[\frac{1}{12n}(n+2)(5n^2-17n+24)R - (n+2)(n-1)\Lambda \right] \\
\text{tr}_{\text{symtr}}(a_2) &= N_2 \left[\frac{1}{360}(n^2-29n+478)R_{MNPQ}R^{MNPQ} \right. \\
&\quad \left. - \frac{1}{360n}(n^3-179n^2+358n+5760)R_{MN}R^{MN} \right. \\
&\quad \left. + \frac{1}{144n^2}(25n^4-95n^3+22n^2+480n+1152)R^2 \right. \\
&\quad \left. + \frac{1}{30n}(n+2)(2n^2-7n+10)\square R \right. \\
&\quad \left. - \frac{1}{6n}(n+2)(5n^2-17n+24)\Lambda R + (n^2+n-2)\Lambda^2 \right]. \tag{2.83}
\end{aligned}$$

The scalar part of the spin-2 lagrangian is given by

$$\frac{1}{e}\mathcal{L}_{EHs} = \frac{1}{2}\phi \left[-\square - \left(\frac{n-4}{n} \right) R + 2\Lambda \right] \phi, \tag{2.84}$$

which, apart from an overall sign, has the same form as eq. (2.10) if we make the substitution $\xi R \rightarrow -\left(\frac{n-4}{n}\right)R + 2\Lambda$. Since the overall sign of Δ contributes a background-field-independent phase to the action which is cancelled by a similar contribution from the ghost action (see below), we may ignore it for the present purposes. With these comments in mind, we may then use the previous results for spin-0 fields to compute the contribution of ϕ to the Gilkey coefficients, a_k .

Finally, we consider the ghosts for the graviton field. Since the gauge-fixing term is $f_N = D^M h_{MN} - \frac{1}{2}D_N h$, and the gauge transformations are $\delta h_{MN} = D_M \xi_N + D_N \xi_M$,

we find the transformation property

$$\delta f_N = \square \xi_N - R^M{}_N \xi_M, \quad (2.85)$$

leading to a complex, fermionic, vector ghost ω_M with lagrangian

$$\frac{1}{e} \mathcal{L} = -\omega_M^* (-\square \delta_N^M + R_N^M) \omega^N. \quad (2.86)$$

The contribution of the vector ghost to the Gilkey coefficients is therefore obtained by multiplying the results found earlier for the real spin-1 field by an overall factor of -2 (and using the choice $\eta = -1$ in eq. (2.22)). We thus obtain the result for the massless graviton in n dimensions (for background Einstein geometries:⁶ $R_{MN} = (R/n) g_{MN}$)

$$\begin{aligned} \text{tr}_2(a_0) &= \frac{N_2}{2} n(n-3) \\ \text{tr}_2(a_1) &= N_2 \left[\frac{1}{12} (5n^2 - 3n + 24)R - n(n+1)\Lambda \right] \\ \text{tr}_2(a_2) &= N_2 \left[\frac{1}{360} (n^2 - 33n + 540)R_{MNPQ}R^{MNPQ} \right. \\ &\quad + \frac{1}{720n} (125n^3 - 497n^2 + 486n - 1440)R^2 \\ &\quad \left. - \frac{n}{6} (5n-7)\Lambda R + n(n+1)\Lambda^2 \right]. \end{aligned} \quad (2.87)$$

Massive Spin 2

We next derive the lagrangian for the massive graviton. In order to do so we require an expression for the quadratic part of the massive spin-2 lagrangian, such as might be obtained from a Kaluza-Klein reduction or as a massive string mode. To keep the analysis as background-independent as possible, we work with the most general such action for which the spin-2 state acquires its mass by mixing with the appropriate Goldstone field, as in the Anderson-Higgs-Kibble mechanism. We believe that by making this requirement we capture quite generally the contributions of the massive spin-2 states which arise in dimensional reduction and as heavy string modes [26].

We start, therefore, with the lagrangian

$$\begin{aligned} \frac{1}{e} \mathcal{L}_{mEH} &= \frac{1}{e} \mathcal{L}_{EH} - \frac{1}{4} F_{MN}F^{MN} - ah^{MN}D_MV_N - bV^MD_Mh \\ &\quad - cR^{MN}V_MV_N - \frac{1}{2}m_1^2 V_MV^M - \frac{1}{2}m_2^2 h_{MN}h^{MN} - \frac{1}{2}\mu_2^2 h^2, \end{aligned} \quad (2.88)$$

⁶We drop $\square R$ in these expressions with only a tiny loss of generality because R is necessarily constant for an Einstein space provided $n > 2$.

where the coefficients a , b , c , m_1 , m_2 and μ_2 are to be determined by demanding the presence of a non-linearly realized gauge symmetry (which would correspond to the diffeomorphisms which do not preserve the background geometry within the Kaluza-Klein context). F_{MN} is the field strength $D_M V_N - D_N V_M$, where we take V_M to have the spin content of a massive spin-1 particle. From the previous sections we see that this should consist of a specific combination of a massless vector field, A_M , and a would-be Goldstone scalar, σ . Accordingly, we make the definition

$$V_M = A_M + p D_M \sigma, \quad (2.89)$$

where the coefficient p is also to be determined in what follows. Notice that, as defined, any lagrangian built from the vector field V_M automatically has the gauge invariance

$$\delta A_M = D_M \epsilon \quad \text{and} \quad \delta \sigma = -\frac{1}{p} \epsilon. \quad (2.90)$$

If we desire we may use unitary gauge for this symmetry to remove σ completely from the theory, however this is not a convenient gauge for our purposes and so in what follows we instead gauge-fix using a more convenient covariant gauge.

In order to implement the underlying gauge invariance which any such a spin-2 field must manifest we ask the above lagrangian to be invariant under the usual spin-2 gauge transformation $\delta h_{MN} = D_M \xi_N + D_N \xi_M$, supplemented by the Goldstone-type transformation $\delta V_M = f \xi_M$. This leads to the following lagrangian⁷

$$\begin{aligned} \frac{1}{e} \mathcal{L}_{mEH} = & \frac{1}{e} \mathcal{L}_{EH} - \frac{1}{4} F_{MN} F^{MN} + f h^{MN} D_M V_N + f V^M D_M h \\ & - R^{MN} V_M V_N - \frac{1}{4} f^2 h_{MN} h^{MN} + \frac{1}{4} f^2 h^2, \end{aligned} \quad (2.91)$$

corresponding to the choices

$$\begin{aligned} a = b = -f, \quad c = 1, \\ m_1 = 0, \\ \text{and} \quad m_2^2 = -\mu_2^2 = \frac{f^2}{2}. \end{aligned} \quad (2.92)$$

⁷As a check on this result, we note that by choosing the gauge where $V_M = 0$, we recover the Pauli-Fierz lagrangian of massive gravity [23]. Also, in flat space, this result agrees (after a suitable field redefinition) with the one given in [24].

We now fix the two gauge freedoms of this action in such a way as to remove the mixings between the various fields having differing spins. To do so we take for the spin-2 gauge-fixing lagrangian

$$\frac{1}{e} \mathcal{L}_{mEH2}^{gf} = - \left(f_N - \frac{1}{2} f V_N \right)^2, \quad (2.93)$$

where f is the parameter appearing in the lagrangian (2.91), and as before f_N is defined as $f_N = D^M h_{MN} - \frac{1}{2} D_N h$. This gauge choice removes the $h^{MN} D_M V_N$ term from the action and introduces a mass term, m , for the vector field, V_M , with $m^2 = \frac{1}{2} f^2$.

To fix the other gauge freedom, eq. (2.90), we add the following gauge-fixing term

$$\frac{1}{e} \mathcal{L}_{mEH1}^{gf} = -\frac{1}{2} (D_M A^M + \lambda h + \rho \sigma)^2, \quad (2.94)$$

with λ and ρ being parameters which are chosen to remove the remaining vector-gravity mixing terms in the quadratic action. In order to do so we again specialize to the case where the background spacetime is an Einstein space, which we also take for simplicity to be a solution to the Einstein equations of the form $G_{MN} + \Lambda g_{MN} = 0$, or $R_{MN} = [2\Lambda/(n-2)] g_{MN}$. Using this we see that the removal of cross terms between A_M , h_{MN} , and σ requires the choices

$$\lambda = -\frac{f}{2}, \quad \text{and} \quad \rho = p q^2, \quad (2.95)$$

where q^2 is defined as

$$q^2 = m^2 + \frac{4\Lambda}{n-2}, \quad (2.96)$$

since in this case the gauge-fixed lagrangian can be written as

$$\mathcal{L}_{mEH} + \mathcal{L}_{mEH1}^{gf} + \mathcal{L}_{mEH2}^{gf} = \mathcal{L}_{mEH0} + \mathcal{L}_{mEH1} + \mathcal{L}_{mEH2}, \quad (2.97)$$

with the decoupled lagrangians, \mathcal{L}_{mEH0} , \mathcal{L}_{mEH1} , and \mathcal{L}_{mEH2} , defined as follows.

\mathcal{L}_{mEH2} denotes the ϕ_{MN} lagrangian, which takes the form

$$\begin{aligned}
\frac{1}{e} \mathcal{L}_{mEH2} &= \frac{1}{e} \mathcal{L}_{EH} - f_N f^N - \frac{1}{4} f^2 h_{MN} h^{MN} + \frac{1}{8} f^2 h^2 \\
&= -\frac{1}{2} \phi_{MN} (\Delta^{MN}_{PQ} + m^2 \bar{\delta}^{MN}_{PQ}) \phi^{PQ} \\
&\quad + \left(\frac{n-2}{4n} \right) h \left(-\square + m^2 + \frac{1}{n} (4-n)R + 2\Lambda \right) h \\
&\quad + \left(\frac{n-4}{n} \right) R_{MN} \phi^{MN} h \\
&= -\frac{1}{2} \phi_{MN} (\Delta^{MN}_{PQ} + m^2 \bar{\delta}^{MN}_{PQ}) \phi^{PQ} \\
&\quad + \frac{1}{2} \phi \left(-\square + m^2 + \frac{4\Lambda}{n-2} \right) \phi,
\end{aligned} \tag{2.98}$$

where ϕ , ϕ_{MN} , $\bar{\delta}^{MN}_{AB}$ and Δ^{MN}_{PQ} are as defined above for the massless spin-2 case. The mass m is related to the symmetry-breaking parameter f by $m^2 = \frac{1}{2} f^2$.

We similarly find the following vector lagrangian, \mathcal{L}_{mEH1} :

$$\begin{aligned}
\frac{1}{e} \mathcal{L}_{mEH1} &= -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} (D^M A_M)^2 - R^{MN} A_M A_N - \frac{1}{2} m^2 A_M A^M \\
&= -\frac{1}{2} A_M \left[(-\square + m^2) \delta_N^M + R_N^M \right] A^N,
\end{aligned} \tag{2.99}$$

where m is the same as for ϕ_{MN} .

Finally, the part of the quadratic action depending on σ is

$$\frac{1}{e} \mathcal{L}_{mEH0} = \frac{1}{2} \left[(p\rho) \sigma \square \sigma - (pf) \sigma \square h - (\rho^2) \sigma^2 + (f\rho) h \sigma \right], \tag{2.100}$$

which contains terms which mix σ and h . However, since p is as yet unspecified we may choose its value to remove these cross terms. This may be done by choosing $p = -f/(4q^2)$ and making the field redefinition $\tilde{\sigma} = \frac{\sqrt{2}m}{4q}(\sigma + 2h)$, after which we find

$$\begin{aligned}
\frac{1}{e} \mathcal{L}_{mEH0} &= -\frac{1}{2} \tilde{\sigma} \left(-\square + m^2 + \frac{4\Lambda}{n-2} \right) \tilde{\sigma} \\
&\quad + \left(\frac{m^2}{4q^2} \right) h \left(-\square + m^2 + \frac{4\Lambda}{n-2} \right) h.
\end{aligned} \tag{2.101}$$

Notice that in this form the last term in \mathcal{L}_{mEH0} (involving h) has the same form as the last term in \mathcal{L}_{mEH2} , and so these can both be combined into \mathcal{L}_{mEH2} by appropriately rescaling the scalar ϕ . Once this is done, and dropping the tilde on σ , the remaining term becomes

$$\frac{1}{e} \mathcal{L}_{mEH0} = -\frac{1}{2} \sigma \left[-\square + m^2 + \frac{4\Lambda}{n-2} \right] \sigma. \tag{2.102}$$

Finally, the action for the ghosts can be easily calculated from the gauge-fixing conditions. The spin-2 gauge-fixing term introduces a complex, fermionic, vector ghost with lagrangian

$$\frac{1}{e} \mathcal{L}_{mEHVgh} = -\omega_M^* \left[(-\square + m^2) \delta_N^M + R_N^M \right] \omega^N. \quad (2.103)$$

Similarly, the spin-1 gauge-fixing term introduces a complex scalar ghost with lagrangian

$$\frac{1}{e} \mathcal{L}_{mEHSgh} = -\omega^* \left(-\square + m^2 + \frac{4\Lambda}{n-2} \right) \omega. \quad (2.104)$$

The complete lagrangian, including all ghosts, for the massive graviton is thus the sum

$$\mathcal{L}_{m2} = \mathcal{L}_{mEH0} + \mathcal{L}_{mEH1} + \mathcal{L}_{mEH2} + \mathcal{L}_{mEHSgh} + \mathcal{L}_{mEHVgh}. \quad (2.105)$$

We are now in a position to assemble the results for a_k . To this end, notice that all fields have been decoupled in the kinetic terms and all now have the same mass, $m^2 = \frac{1}{2}f^2$. This allows us to sum the separate contributions to a_k from each of these fields. It is also interesting to note that the scalar fields h , σ and the complex scalar ghost all have precisely the same lagrangian, and so their net effect is to completely cancel one another in the one-loop action. Similarly, the vector boson A_M and the complex vector ghost also share the same lagrangian, and so for our purposes these two together contribute the equivalent of one real vector ghost.

In summary, the one-loop divergences for the massive graviton are given by the sum of the divergences of a symmetric traceless field and one real vector ghost (for which $\eta = -1$). Thus, we find

$$\begin{aligned} \text{tr}_{2m}(a_0) &= \frac{N_2}{2}(n+1)(n-2) \\ \text{tr}_{2m}(a_1) &= N_2 \left[\frac{(6-n)(n+4)(n+1)\Lambda}{6(n-2)} \right] \\ \text{tr}_{2m}(a_2) &= N_2 \left[\frac{1}{360}(n^2 - 31n + 508)R_{MNPQ}R^{MNPQ} \right. \\ &\quad \left. + \frac{(5n^4 - 7n^3 - 248n^2 - 596n - 1440)\Lambda^2}{180(n-2)^2} \right] \end{aligned} \quad (2.106)$$

for n -dimensional massive gravitons on background metrics satisfying $G_{MN} + \Lambda g_{MN} = 0$.

3. Supergravity Models

In supergravity theories the ultraviolet sensitivity of the low-energy theory is often weaker than in non-supersymmetric models. This weaker sensitivity arises due to cancellations between the effects of bosons and fermions in loops. The purpose of this section is to illustrate the utility of the previous section's results by using them to exhibit this cancellation explicitly for supergravities in various dimensions. Some of the results we obtain — particularly those for massless particles in higher-dimensional supergravities — are computed elsewhere, and we use the agreement between these earlier calculations and our results as a check on the validity of our computations.

We proceed by summing the above expressions over the particles appearing in the appropriate supermultiplets. The result for the ultraviolet-sensitive part of the one-loop action obtained by integrating out a supermultiplet is given by

$$\Sigma_{UV} = \frac{1}{2} \left(\frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} \sum_p (-)^{F(p)} m_p^{n-2k} \Gamma(k - n/2) \text{tr}_p[a_k], \quad (3.1)$$

where the sum on p runs over the elements of a supermultiplet. As is clear from this expression, it is the weighted sum $\sum_p (-)^{F(p)} m_p^{n-2k} \text{tr}_p[a_k]$ which is of interest in supersymmetric theories.

In Minkowski space the strongest suppression of UV sensitivity arises when supersymmetry is unbroken, in which case all members of a supermultiplet share the same mass (so that $m_p = m$ for all p). In this case, eq. (3.1) can be written as

$$\Sigma_{UV} = \frac{1}{2} \left(\frac{1}{4\pi} \right)^{n/2} \int d^n x \sqrt{-g} \sum_{k=0}^{[n/2]} m^{n-2k} \Gamma(k - n/2) \text{Tr}[a_k], \quad (3.2)$$

where

$$\text{Tr}[a_k] \equiv \sum_p (-)^{F(p)} \text{tr}_p[a_k] \quad (3.3)$$

is the relevant combination of heat-kernel coefficients for a supermultiplet. Since $\text{tr}[a_0]$ simply counts the spin states of the corresponding particle type, the cancellation of the leading UV sensitivity occurs for a mass-degenerate supermultiplet simply because each supermultiplet contains equal numbers of bosons and fermions:

$$\text{Tr}[a_0] = \sum_p (-)^{F(p)} \text{tr}_p[a_0] = N_B - N_F = 0. \quad (3.4)$$

This ensures the absence of a dependence of the form m^n in Σ_{UV} .

The story is more complicated when there is a nonzero cosmological constant, and this is due to the fact that mass itself is more delicate to define in de Sitter or anti-de Sitter spacetimes. For Minkowski space mass can be defined for particle states as a Casimir invariant of the Poincaré group, but this definition is no longer appropriate when Λ is nonzero because Poincaré transformations are then not the relevant spacetime isometries. Rather, for de Sitter space the relevant isometry group in four dimensions is $\text{SO}(4, 1)$, while the isometries of anti-de Sitter space fill out the group $\text{SO}(3, 2)$. For these geometries it only makes sense to inquire about the implications of unbroken supersymmetry for the anti-de Sitter case. This is because supersymmetry is always broken in de Sitter spacetime, whereas there is a supersymmetric generalization of $\text{SO}(3, 2)$ for which one can find particle supermultiplets which represent the unbroken supersymmetry.

In our previous calculations of the Gilkey coefficients we have defined m^2 to be that piece in the operator $(-\square + X)$ which is a constant for *arbitrary* background fields.⁸ We nevertheless must still grapple with the above ambiguities as to the meaning of mass in de Sitter and anti-de Sitter spacetimes, due to the freedom of absorbing into m^2 contributions coming from the background curvature for constant-curvature spacetimes. One can try to restrict this freedom by demanding masslessness to correspond to conformal invariance or (for higher-spin fields) to unbroken gauge invariance, bearing in mind that these choices need not imply propagation along the light cone [25].

The upshot of this discussion is that it need not be true that all of the particles within a supermultiplet share the same mass even when working about a supersymmetric AdS background. In such cases one cannot pull a common mass out of the sum over particles within a supermultiplet, as was done in going from eq. (3.1) to eq. (3.2).

To see this concretely, consider the specific example of a Wess-Zumino multiplet in $n = 4$ spacetime dimensions expanded about a supersymmetric AdS background. Such a multiplet consists of a scalar, pseudoscalar, and spinor field: (S, P, χ) , and taking the scalar and pseudoscalar to have a conformal coupling parameter, $\xi = -1/6$, their mass terms can be written as $m_S^2 = m^2 - \delta m^2$, $m_P^2 = m^2 + \delta m^2$ and $m_\chi^2 = m^2$. Unbroken supersymmetry implies that these mass terms are related to one another by $m^2 = \mu^2 \Lambda / 12$ and $\delta m^2 = \mu \Lambda / 6$, where Λ is the AdS cosmological

⁸This statement requires appropriate modification in the case of spin 2, where we include a cosmological constant term in the Lagrangian.

constant (which is positive in our conventions) and μ is a dimensionless parameter which classifies the massive supersymmetric particle representations. In this case, we find

$$\begin{aligned} \sum_p (-)^{F(p)} m_p^4 \text{tr}_p[a_0] &= m_S^4 \text{tr}_S[a_0] + m_P^4 \text{tr}_P[a_0] - m_\chi^4 \text{tr}_\chi[a_0] = 2\delta m^4 = \frac{\mu^2 \Lambda^2}{18} \\ \sum_p (-)^{F(p)} m_p^2 \text{tr}_p[a_1] &= m_S^2 \text{tr}_S[a_1] + m_P^2 \text{tr}_P[a_1] - m_\chi^2 \text{tr}_\chi[a_1] = -\frac{2m^2 \Lambda}{3} = -\frac{\mu^2 \Lambda^2}{18} \\ \sum_p (-)^{F(p)} m_p^0 \text{tr}_p[a_2] &= \text{tr}_S[a_2] + \text{tr}_P[a_2] - \text{tr}_\chi[a_2] = \frac{R_{MNPQ}^2}{48} - \frac{\Lambda^2}{9}. \end{aligned} \quad (3.5)$$

The above complication keeps us from quoting general expressions for the sum of the Gilkey coefficients over arbitrary supermultiplets in general dimensions, since for AdS backgrounds these must be computed with the specific dependence of the relevant masses on Λ . Notice however that last expression in eq. (3.5) contains no dependence on the individual particle masses (since $\text{tr}_p[a_2]$ is multiplied by $m_p^0 = 1$). Terms which are only present in the mass invariant piece of Σ_{UV} , such as R_{MNPQ}^2 and F_{MN}^2 , can be calculated once and for all in a model-independent way because their coefficients do not depend on the details of the particle masses involved. This we do in Tables (12) and (15) for various 4D supermultiplets. As can be seen from the above example, however, calculating the complete answer for Σ_{UV} is not difficult once the individual particle masses are known. Similar considerations hold for dimensions other than four, with some terms in Σ_{UV} being mass independent and others requiring more detailed knowledge of the particle spectrum about a given background.

Equations of Motion

In the remainder of this section we use the previous results to compute the statistics-weighted sum of $\text{tr}[a_1]$ and $\text{tr}[a_2]$ over the particle content obtained by linearizing various supergravity theories about different solutions to their classical field equations. To this end we must evaluate results of the previous section at the solutions to the relevant field equations,⁹ and sum over the relevant particle content describing these fluctuations.

The field equations for a very broad class of supergravities become reasonably simple once restricted only to background metrics, gauge fields and the scalar dilaton.

⁹Recall that we are always free to use the classical equations of motion to simplify any one-loop quantity (like Σ_{UV}), because any one-loop term which vanishes with the classical field equations may be removed from Σ by performing an appropriate field redefinition [1, 2].

These equations may be derived from the action

$$S = - \int d^n x \sqrt{-g} \left[\frac{1}{2} R + \frac{1}{2} \partial_M \phi \partial^M \phi + V(\phi) + \frac{1}{4} e^{\lambda \phi} F_{MN}^a F_a^{MN} + \frac{1}{2r!} e^{\beta \phi} H_{M_1 \dots M_r} H^{M_1 \dots M_r} \right], \quad (3.6)$$

where λ and β are dimension- and supergravity-dependent numbers and V is a dimension- and supergravity-dependent potential for the dilaton ϕ . Notice that we use units here for which Newton's constant satisfies $\kappa = 1$.

The simplest class of solutions to these equations are those for which the gauge fields vanish, $F_{MN}^a = 0$, and the dilaton is constant, $\partial_M \phi = 0$, at a value for which $V' = 0$. (More general solutions having nonzero background gauge fields, F_{MN}^a , are also possible and usually — but not always — require a non-constant background dilaton configuration as well: $\partial_M \phi \neq 0$.) In this case the field equations require the metric to be an Einstein space, $G_{MN} + \Lambda g_{MN} = 0$, or

$$R_{MN} = \left(\frac{2\Lambda}{n-2} \right) g_{MN}, \quad (3.7)$$

where $\Lambda = -V$, evaluated at the vacuum configuration.

3.1 11D Example

Eleven-dimensional supergravity has a particularly simple field content, consisting of a vielbein (or metric), a gravitino, and an antisymmetric 3-form, and so provides a simple starting example. Our purpose in this example is to compare with the known results of ref. [10] as a check on our calculations.¹⁰ The contributions to some of the Gilkey coefficients specialized to 11 dimensions are listed in Table (1).

Because the theory has equal numbers of bosons and fermions, we have $\text{Tr}(a_0) = 0$. Because the background metric is Ricci flat, it also follows that $\text{Tr}(a_1) = 0$ and $\text{Tr}(a_2) \propto R_{MNPQ} R^{MNPQ}$. Summing the coefficients in Table (1) then shows that

$$\text{Tr}_{11D}(a_2) = \sum_p (-)^{F(p)} \text{tr}_p(a_2) = \frac{1}{180} [219 - 368 + 149] R_{MNPQ}^2 = 0, \quad (3.8)$$

in agreement with ref. [10].

¹⁰For the case of the graviton, the terms we find proportional to the Ricci scalar appear to differ with those of [10]. However there is no discrepancy once we specialize to solutions of the equations of motion because their analysis assumes that $\Lambda = 0$, and so their classical equations of motion require $R = 0$.

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | |
|---------------|--------------------------|--------------------------|---------------------------|--------------------|
| | 1 | $\frac{1}{3}R$ | $\frac{1}{180}R_{MNPQ}^2$ | $\frac{1}{495}R^2$ |
| a/s 3-form | 84 | 21 | 219 | 84 |
| gravitino (M) | -128 | -32 | -368 | -188 |
| graviton | 44 | 149 | 149 | 6884 |

Table 1: Gilkey coefficients for massless states in 11D, using $R_{MN} = (R/n)g_{MN}$.

The same result can also be obtained for geometries of the form $\mathcal{M}_6 \times T_5$ without having to use expressions for the contribution of a 3-form field, simply by truncating the 11D theory to 6D, such as would be obtained for the massless Kaluza-Klein spectrum by dimensionally reducing on a 5-torus [10]. The 6D spectrum obtained in this way consists of: 1 graviton, 4 Weyl gravitini, 5 2-form potentials, 16 (1-form) gauge fields, 20 Weyl fermions, and 25 scalars which we take to be minimally coupled. Since in 6 dimensions a 3-form potential is dual to a 1-form, the entire dimensionally-reduced field content can be handled using the expressions given above. Summing the 6D results — given explicitly in Table (4) below — for this field content, and specializing to the case of a Ricci-flat 6D metric (with all gauge field strengths vanishing, $F_{\mu\nu} = 0$), again gives the results $\text{Tr } (a_0) = \text{Tr } (a_1) = \text{Tr } (a_2) = 0$.

3.2 10D Examples

The supergravities of interest in 10 dimensions are those which arise as the low-energy limits of heterotic, Type I, Type IIA and Type IIB string theories. Since results for the Gilkey coefficients are known for each of these, we briefly consider them in turn. For convenience, the specialization of the previous sections' formulae to the case $n = 10$ is given in Table (2). (This table is also specialized to the choices $C(\mathcal{R}_0) = C(\mathcal{R}_{3/2}) = 0$, as is appropriate for these 10D supergravities.)

Type IIA and IIB Theories

The field content of the Type IIA theory is given by the metric, g_{MN} , two Majorana-Weyl gravitini having opposite chiralities, ψ_M^r , a 3-form gauge potential, C_{MNP} , a 2-form potential, B_{MN} , a gauge potential, C_M , two Majorana-Weyl dilatini (with opposite chiralities), χ^r , plus a dilaton, ϕ . The dilaton potential vanishes, $\Lambda = -V = 0$.

Summing the contributions of each field to the Gilkey coefficients, and evaluating at Ricci-flat metrics with vanishing gauge potentials again gives the result $\text{Tr } (a_0) =$

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | |
|-------------------------|--------------------------|--------------------------|---------------------------|-----------------------------|
| | 1 | $\frac{1}{6}R$ | $\frac{1}{180}R_{MNPQ}^2$ | $\frac{1}{12}g_a^2F_{MN}^2$ |
| spin zero ($\xi = 0$) | 1 | -1 | 1 | - |
| spin half (M-W) | -8 | -4 | 7 | $-16C(\mathcal{R}_{1/2})$ |
| spin one | 8 | -2 | -7 | $16C(A)$ |
| a/s 2-form | 28 | 8 | 28 | - |
| a/s 3-form | 56 | 34 | 191 | - |
| a/s 4-form | 70 | 50 | 310 | - |
| gravitino (M-W) | -56 | -28 | -191 | - |
| graviton | 35 | 247 | 155 | - |

Table 2: Gilkey coefficients for massless states in 10D. Terms in a_2 involving only the Ricci tensor or Ricci scalar are not explicitly displayed. Hyphens indicate quantities which do not arise and so are not tabulated.

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | |
|-------------------------|--------------------------|--------------------------|---------------------------|-----------------------------|
| | 1 | $\frac{1}{30}R$ | $\frac{1}{180}R_{MNPQ}^2$ | $\frac{1}{12}g_a^2F_{MN}^2$ |
| spin zero ($\xi = 0$) | 1 | -5 | 1 | $-C(\mathcal{R}_0)$ |
| spin one-half (M) | -16 | -40 | 14 | $-32C(\mathcal{R}_{1/2})$ |
| spin one | 9 | -15 | -6 | $15C(A)$ |
| a/s 2-form | 36 | 30 | 21 | - |
| a/s 3-form | 84 | 210 | 219 | - |
| a/s 4-form | 126 | 420 | 501 | - |
| gravitino (M) | -128 | -320 | -368 | $-256C(\mathcal{R}_{3/2})$ |
| graviton | 44 | 1142 | 149 | - |

Table 3: Gilkey coefficients for massive states in 10D. Terms in a_2 involving only the Ricci tensor or Ricci scalar are not explicitly displayed. Hyphens indicate quantities which do not arise and so are not tabulated.

$\text{Tr } (a_1) = 0$ and

$$\text{Tr}_{IIA}(a_2) = \frac{1}{180} \left[155 - 2(191) + 191 + 28 - 7 + 2(7) + 1 \right] R_{MNPQ}^2 = 0. \quad (3.9)$$

This may also be understood using the vanishing of these quantities in 11 dimensions because the Type IIA theory can be obtained by dimensionally reducing the 11D theory on a circle.

The field content of the Type IIB theory is obtained from the Type IIA theory

by giving the fermions the same – rather than opposite – chirality and by replacing the 1- and 3-form potentials by a scalar (0-form), C , a 2-form, C_{MN} , and a self-dual 4-form, C_{MNPQ} . For this theory the dilaton potential again vanishes so $\Lambda = -V = 0$. The statistics-weighted sum of the Gilkey coefficients a_0 , a_1 and a_2 again vanishes for this field content, as may be seen since

$$\mathrm{Tr}_{IIB}(a_2) = \frac{1}{180} \left[155 - 2(191) + \frac{1}{2}(310) + 2(28) + 2(7) + 2(1) \right] R_{MNPQ}^2 = 0. \quad (3.10)$$

Part of this result can again be understood in a different way, since the Type IIA and IIB supergravities produce the same theory when dimensionally reduced on a 2-torus to 9 dimensions. Since we know from the above that the Type IIA theory gives $\mathrm{Tr}(a_0) = \mathrm{Tr}(a_1) = \mathrm{Tr}(a_2) = 0$ for this kind of compactification, it follows that these quantities must also vanish for Type IIB theories when evaluated on a 9-dimensional Ricci-flat background.

Heterotic and Type I Theories

The field content of the Type I and heterotic theories consist of a 10D $N = 1$ supergravity multiplet coupled to a 10D super-Yang-Mills multiplet for the gauge groups $E_8 \times E_8$ or $SO(32)$, both of which are 496-dimensional.

The $N = 1$ supergravity multiplet in 10D consists of: one graviton g_{MN} , one Majorana-Weyl gravitino ψ_M , one 2-form potential B_{MN} , one Majorana-Weyl spin-1/2 fermion χ and a scalar dilaton ϕ . For Type I and heterotic models the 10D gauge multiplet consists of N_A gauge fields A_M^a and N_A Majorana-Weyl spinors λ^a , where $N_A = 496$ is the dimension of the gauge group. These supergravities have vanishing dilaton potential, $V = \Lambda = 0$, but are distinguished from one another by the value of the gauge-dilaton coupling, which is given by $\lambda = -4/(n - 2) = -1/2$ for the heterotic theory, or $\lambda = (n - 6)/(n - 2) = +1/2$ for the Type I theory.

Specializing to backgrounds with vanishing gauge fields and constant dilaton field leads to vacuum space-times for which $R_{MN} = 0$. It is then simple to see that the contributions to the Gilkey coefficients of the gauge supermultiplet vanishes, with the coefficients of the R_{MNPQ}^2 and F_{MN}^2 terms both cancelling between the gauge bosons and the gauginos. For the gravity supermultiplet in these theories we also trivially have $\mathrm{Tr}(a_0) = \mathrm{Tr}(a_1) = 0$ and

$$\mathrm{Tr}_{I,\text{het}}(a_2) = \frac{1}{180} \left[155 - 191 + 28 + 7 + 1 \right] R_{MNPQ}^2 = 0, \quad (3.11)$$

again in agreement with ref. [10].

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | | | $(-)^F \text{tr } (a_2) _{ms}$ |
|----------------------------|--------------------------|--------------------------|---------------------------|--------------------------|------------------------------|--------------------------------|
| | 1 | $\frac{1}{10}\Lambda$ | $\frac{1}{360}R_{MNPQ}^2$ | $\frac{1}{600}\Lambda^2$ | $\frac{1}{12}g_a^2 F_{MN}^2$ | $\frac{1}{25}\Lambda^2$ |
| spin zero ($\xi = 0$) | 1 | -5 | 2 | 70 | $-C(\mathcal{R}_0)$ | 3 |
| spin zero ($\xi = -1/5$) | 1 | 1 | 2 | -2 | $-C(\mathcal{R}_0)$ | 0 |
| spin one-half (W) | -4 | -10 | 7 | -55 | $-8C(\mathcal{R}_{1/2})$ | -2 |
| spin one | 4 | 10 | -22 | -170 | $20C(A)$ | -8 |
| a/s 2-form | 6 | 30 | 132 | 420 | - | 23 |
| gravitino (W) | -12 | -42 | -219 | 1419 | - | 50 |
| graviton | 9 | 45 | 378 | -2970 | - | -108 |

Table 4: 6D Results for Massless Fields, computed using $R_{MN} = \frac{1}{2}\Lambda g_{MN}$. The last column gives the result if the spacetime is also maximally symmetric in 6 dimensions: $R_{MNPQ} = (\Lambda/10)(g_{MP}g_{NQ} - g_{NP}g_{MQ})$.

3.2.1 Massive 10D Fields

Massive 10D fields can arise in two ways in string theory. They can arise as KK modes in the dimensional reduction of 11D supergravity on a circle or a line segment, or as massive string modes within the usual 10D string theories. Indeed, these two ways are famously believed to be equivalent [27]. The contributions to the heat-kernel coefficients from various massive 10D fields are listed in Table (3).

A simple example which uses these results is the contribution of a massive KK level which arises when the 11D theory is compactified down to 10D on a circle. Writing the 10D indices as $\mu = 0, \dots, 9$ and the 11th index as s , the 10D field content obtained by dimensionally reducing in this case consists of the metric components ($g_{\mu\nu}$, $g_{\mu s}$ and g_{ss}); the gravitino components (ψ_μ and ψ_s); and the 3-form components ($C_{\mu\nu\lambda}$ and $C_{\mu\nu s}$). From the results of the previous sections we see that these have the same field content as a single massive 10D spin-2 particle, a single massive 10D spin-3/2 particle and a single massive 3-form potential, and so

$$\text{Tr}_{10D-KK}(a_2) = \frac{1}{180} [149 - 368 + 219] R_{MNPQ}^2 = 0. \quad (3.12)$$

3.3 6D Examples

In 6 dimensions there is a larger variety of supergravity theories possible than in 10 dimensions, and so in this case we present our results in terms of the various supermultiplets which are encountered rather than attempting to independently list

| Multiplet | Particle Content | Number of States |
|-----------|--|------------------|
| Hyper | 2 spin 0 + 1 (symp-W) spin 1/2 | $2_B + 2_F$ |
| Gauge | 1 spin 1 + 2 (symp-W) spin 1/2 | $4_B + 4_F$ |
| Tensor | 1 spin 0 + 2 (symp-W) spin 1/2 + 1 (anti) self-dual 2-form | $4_B + 4_F$ |
| Gravitino | 1 (symp-W) spin 1/2 + 2 spin 1 + 1 (symp-W) spin 3/2 | $8_B + 8_F$ |
| Graviton | 1 self-dual 2-form + 2 (symp-W) spin 3/2 + 1 spin 2 | $12_B + 12_F$ |

Table 5: Particle content of massless 6D supermultiplets.

| | Tr (a_2) | |
|-----------|--------------------------|----------------------------|
| | $\frac{1}{48}R_{MNPQ}^2$ | $\frac{1}{8}g_a^2F_{MN}^2$ |
| Hyper | 1 | $-4C(\mathcal{R}_h)$ |
| Gauge | -2 | $8C(A)$ |
| Tensor | 10 | - |
| Gravitino | -20 | $16C(\mathcal{R}_{3/2})$ |
| Gravity | 30 | - |

Table 6: 6D results for massless supermultiplets, assuming $\Lambda = 0$. It is assumed that the tensor and graviton multiplets do not carry the charge to which the background gauge fields couple.

the most commonly-occurring of the supergravities which are possible. Since the particle content of a supermultiplet depends on whether or not the particles are massless or massive, we treat each separately. Since we allow $\Lambda \neq 0$ in Tables (4)

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | | | $(-)^F \text{tr } (a_2) _{ms}$ |
|----------------------------|--------------------------|--------------------------|---------------------------|--------------------------|-----------------------------|--------------------------------|
| | 1 | $\frac{1}{10}\Lambda$ | $\frac{1}{360}R_{MNPQ}^2$ | $\frac{1}{600}\Lambda^2$ | $\frac{1}{12}g_a^2F_{MN}^2$ | $\frac{1}{25}\Lambda^2$ |
| spin zero ($\xi = 0$) | 1 | -5 | 2 | 70 | $-C(\mathcal{R}_0)$ | 3 |
| spin zero ($\xi = -1/5$) | 1 | 1 | 2 | -2 | $-C(\mathcal{R}_0)$ | 0 |
| spin one-half (symp) | -4 | -10 | 7 | -55 | $-8C(\mathcal{R}_{1/2})$ | -2 |
| spin one | 5 | 5 | -20 | -100 | $19C(A)$ | -5 |
| a/s 2-form | 10 | 40 | 110 | 250 | - | 15 |
| gravitino (symp) | -16 | -80 | -212 | -220 | $-32C(\mathcal{R}_{3/2})$ | -18 |
| graviton | 14 | 0 | 358 | -1870 | - | -63 |

Table 7: 6D Results for Massive Fields, with $R_{MN} = \frac{1}{2}\Lambda g_{MN}$. The last column gives the result if the 6 dimensions are maximally symmetric.

| | Multiplet | Field Equivalent |
|-----------|-------------------------------|--|
| Gauge | 16 _{<i>m</i>} | $(A_M^m, 2\psi^m, 3\phi^m)$ |
| Gravitino | 64 _{<i>m</i>} | $(\psi_M^m, 2A_{MN}^m, 2A_M^m, 4\psi^m, 2\phi^m)$ |
| Gravity | 80 _{<i>m</i>} | $(g_{MN}^m, 2\psi_M^m, A_{MN}^m, 3A_M^m, 2\psi^m, \phi^m)$ |

Table 8: Massive representations of $(2, 0)$ supersymmetry in 6 dimensions, labelled by their dimension. Note that the fermions in this table are not chiral and the 2-form potentials are not self-dual or anti-self-dual. The superscript ‘*m*’ indicates the corresponding field describes a massive particle (rather than massless).

| | Tr (a_2) | |
|-----------|--------------|------------|
| | R_{MNPQ}^2 | F_{MN}^2 |
| Gauge | 0 | 0 |
| Gravitino | 0 | — |
| Gravity | 0 | — |

Table 9: 6D results for massive supermultiplets, assuming $\Lambda = 0$. It is assumed that the tensor and graviton multiplets do not carry the charge to which the background gauge fields couple.

and (7), in tabulating the Gilkey coefficients for the gravitino we use the results from the appendix.

3.3.1 Massless Multiplets

The contributions to the Gilkey coefficients which result for massless particles in 6 dimensions are listed in Table (4), and the field content of the commonly occurring massless supermultiplets for 6D supersymmetry are listed in Table (5). For the case $\Lambda = 0$, the resulting nonzero heat-kernel coefficients for these multiplets are given in Table (6). In this table we imagine that all of the particles in a given supermultiplet share the same charge for the background gauge fields, which is true if the relevant gauge symmetries commute with supersymmetry. Because of this choice we also take the 2-form, gravitino and gauge fields to be neutral under the gauge symmetry.

3.3.2 Massive Multiplets

For massive 6D particles, the contributions to the Gilkey coefficients found from the previous section are listed in Table (7). The field content of the commonly-

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | | | $(-)^F \text{tr } (a_2) _{ms}$ |
|----------------------------|--------------------------|--------------------------|---------------------------|-------------------------|------------------------------|--------------------------------|
| | 1 | $\frac{1}{3}\Lambda$ | $\frac{1}{720}R_{MNPQ}^2$ | $\frac{1}{45}\Lambda^2$ | $\frac{1}{12}g_a^2 F_{MN}^2$ | $\frac{1}{270}\Lambda^2$ |
| spin zero ($\xi = 0$) | 1 | -2 | 4 | 9 | $-C(\mathcal{R}_0)$ | 58 |
| spin zero ($\xi = -1/6$) | 1 | 0 | 4 | -1 | $-C(\mathcal{R}_0)$ | -2 |
| spin one-half (M) | -2 | -2 | 7 | -3 | $-4C(\mathcal{R}_{1/2})$ | -11 |
| spin one | 2 | 8 | -52 | -12 | $22C(A)$ | -124 |
| a/s 2-form | 1 | -2 | 364 | 9 | - | 418 |
| gravitino (M) | -2 | -18 | -233 | 137 | $-4C(\mathcal{R}_{3/2})$ | 589 |
| graviton | 2 | 32 | 848 | -522 | - | -2284 |

Table 10: 4D Results for Massless Fields, with $R_{MN} = \Lambda g_{MN}$. The last column specializes to the maximally-symmetric case, which in 4D implies $R_{MNPQ} = (\Lambda/3)(g_{MP}g_{NQ} - g_{NP}g_{MQ})$.

occurring massive supermultiplets for 6D supersymmetry are also listed in Table (8). The resulting heat-kernel coefficients for these multiplets are then given in Table (9) for the case $\Lambda = 0$. In this table we imagine that all of the particles in a given supermultiplet share the same charge for the background gauge fields, which is true if the relevant gauge symmetries commute with supersymmetry. Because of this choice we also take the 2-form, gravitino and gauge fields to be neutral under the gauge symmetry. Notice, in particular, how $\text{Tr } (a_2)$ vanishes for these 6D massive multiplets provided the backgrounds are Ricci-flat ($\Lambda = 0$), as reported in a companion paper [12].

3.4 4D Examples

There are considerably more supergravity theories possible in 4 dimensions than 6, and so we again list results as a function of the particle content of 4D supermultiplets. As for the 6D case this requires a separate discussion of the massless and massive cases. A summary of the results of previous sections, specialized to Einstein geometries, $R_{MN} = \Lambda g_{MN}$ is given in Table (10). Since we allow Λ to be nonzero, we use the results in the appendix to tabulate the Gilkey coefficients for the gravitino.

3.4.1 Massless Multiplets

The field content of the usual massless supermultiplets for 4D supersymmetry are listed in Table (11). The corresponding nonzero heat-kernel coefficients for these multiplets are given in Table (12) for the case $\Lambda = 0$. If there is a nonzero cosmological constant, then as discussed at the beginning of this section, there can be additional

| Multiplet | Particle Content | Number of States |
|-----------|---------------------------|------------------|
| Matter | 2 spin 0 + 1 (W) spin 1/2 | $2_B + 2_F$ |
| Gauge | 1 spin 1 + 1 (W) spin 1/2 | $2_B + 2_F$ |
| Gravitino | 1 spin 1 + 1 (W) spin 3/2 | $2_B + 2_F$ |
| Gravity | 1 (W) spin 3/2 + 1 spin 2 | $2_B + 2_F$ |

Table 11: Particle content for $N = 1$ massless supermultiplets in 4D.

| | Tr (a_2) | |
|-----------|--------------------------|----------------------------|
| | $\frac{1}{48}R_{MNPQ}^2$ | $\frac{1}{2}g_a^2F_{MN}^2$ |
| Matter | 1 | $-C(\mathcal{R}_m)$ |
| Gauge | -3 | $3C(A)$ |
| Gravitino | -19 | - |
| Gravity | 41 | - |

Table 12: Results for massless supermultiplets in 4D. For the case $\Lambda \neq 0$, there will be additional Λ -dependent terms which we do not write.

Λ -dependent terms. We imagine that all of the particles in a given supermultiplet share the same charge for the background gauge fields. As usual we also take the 2-form, gravitino and skew-tensor fields to be neutral under the background gauge symmetry.

Although the contributions of 4D multiplets are typically nonzero, they often give zero once they are summed over the particle content of a multiplet of extended supersymmetry. For example, combining one gauge multiplet with 3 conformally-coupled ($\xi = -\frac{1}{6}$) matter multiplets in the adjoint representation ($\mathcal{R}_m = A$) gives the field content of $N = 4$ super-Yang Mills theories. Specializing to flat space ($\Lambda = 0$) and summing the appropriate entries in Table (12) then reproduces the famous result $\text{Tr}(a_0) = \text{Tr}(a_1) = \text{Tr}(a_2) = 0$ for this combination.

3.4.2 Massive Multiplets

Finally, the heat-kernel coefficients for massive 4D fields are given in Table (13). These results may then be assembled into massive representations of 4D supersymmetry, as listed in Table (14). The corresponding heat-kernel coefficients for these multiplets are given in Table (15), assuming all members of the multiplet share the same mass (i.e., assuming $\Lambda = 0$). Again, if $\Lambda \neq 0$, then there can be additional

| | $(-)^F \text{tr } (a_0)$ | $(-)^F \text{tr } (a_1)$ | $(-)^F \text{tr } (a_2)$ | | | $(-)^F \text{tr } (a_2) _{ms}$ |
|----------------------------|--------------------------|--------------------------|---------------------------|-------------------------|-----------------------------|--------------------------------|
| | 1 | $\frac{1}{3}\Lambda$ | $\frac{1}{720}R_{MNPQ}^2$ | $\frac{1}{45}\Lambda^2$ | $\frac{1}{12}g_a^2F_{MN}^2$ | $\frac{1}{270}\Lambda^2$ |
| spin zero ($\xi = 0$) | 1 | -2 | 4 | 9 | $-C(\mathcal{R}_0)$ | 58 |
| spin zero ($\xi = -1/6$) | 1 | 0 | 4 | -1 | $-C(\mathcal{R}_0)$ | -2 |
| spin one-half (M) | -2 | -2 | 7 | -3 | $-4C(\mathcal{R}_{1/2})$ | -11 |
| spin one | 3 | 6 | -48 | -3 | $21C(A)$ | -66 |
| a/s 2-form | 3 | 6 | 312 | -3 | - | 294 |
| gravitino (M) | -4 | -16 | -226 | 24 | $-8C(\mathcal{R}_{3/2})$ | -82 |
| graviton | 5 | 20 | 800 | -435 | - | -1810 |

Table 13: 4D Results for massive fields, with $R_{MN} = \Lambda g_{MN}$. The last column specializes to maximally-symmetric 4D background geometries.

| Multiplet | Particle Content | Number of States |
|-----------|--|------------------|
| Matter | 2 spin 0 + 1 (M) spin 1/2 | $2_B + 2_F$ |
| Gauge | 1 spin 0 + 2 (M) spin 1/2 + 1 spin 1 | $4_B + 4_F$ |
| Gravitino | 1 (M) spin 1/2 + 2 spin 1 + 1 (M) spin 3/2 | $6_B + 6_F$ |
| Gravity | 1 spin 1 + 2 (M) Spin 3/2 + 1 spin 2 | $8_B + 8_F$ |

Table 14: Particle content for massive $N = 1$ supermultiplets in 4D.

| | Tr (a_2) | |
|-----------|--------------------------|----------------------------|
| | $\frac{1}{48}R_{MNPQ}^2$ | $\frac{1}{2}g_a^2F_{MN}^2$ |
| Matter | 1 | $-C(\mathcal{R}_m)$ |
| Gauge | -2 | $2C(A)$ |
| Gravitino | -21 | - |
| Gravity | 20 | - |

Table 15: Results for massive supermultiplets in 4D. For the case $\Lambda \neq 0$, there will be additional Λ -dependent terms which we do not write.

Λ -dependent terms which we do not follow. We take all of the particles in a given supermultiplet to share the same charge for the background gauge fields. The 2-form, gravitino and skew-tensor fields are taken to be neutral under the background gauge symmetry.

4. Conclusions

This paper accomplishes several aims regarding one-loop contributions to the effective action for a wide class of field theories in a variety of dimensions.

First, we set up the quadratic part of the action for spins 0 through 2 in arbitrary spacetime dimensions in a way which is useful for calculations. In particular, we set up a covariant gauge for each spin which removes all mixings between fields that transform differently under local Lorentz transformations. For massive particles we show how to disentangle the higher-spin fields from their lower-spin would-be Goldstone counterparts.

We then use this formulation to compute the leading ultraviolet sensitivity which arises within a loop of any such particle. We are able to do so because the gauge choice described above allows us to use standard results for the heat-kernel coefficients for a broad class of background fields. Finally, we tabulate these coefficients for some of the fields and dimensions (4,6,10 and 11) of particular interest for applications.

We expect the generality of our expressions to be useful for a variety of future applications.

Acknowledgements

We thank Arkady Tseytlin for helpful discussions. This research was supported in part by a McGill University graduate fellowship as well as research funds from N.S.E.R.C. (Canada), McMaster University and from the Killam Foundation.

A. Appendix: Gravitini With $\Lambda \neq 0$

In this appendix we slightly generalize the treatment of massless and massive spin-3/2 particles given in the main text to include the possibility that the lagrangian density includes a nonzero cosmological constant (or a nontrivial scalar potential once the background scalar field equations are satisfied). As discussed in §3, the nonzero cosmological constant implies particles in a supermultiplet need no longer be degenerate in mass, and so we calculate here how this effect plays out for the gravitino. For instance, this case arises in four dimensions, where an anti-de Sitter (AdS) cosmological constant term in the action is not precluded by supersymmetry itself. Even though the application of most interest is to four dimensions, we carry the

spacetime dimension n as a variable in this appendix in case more general applications of the expressions derived here should become of interest.

Massless Gravitino

In this case we take the spin-2 field to be described by the Einstein-Hilbert action supplemented by the cosmological term, which in our conventions is

$$\frac{1}{e} \mathcal{L}_{EH} = -\frac{1}{2\kappa^2} (R - 2\Lambda). \quad (\text{A.1})$$

Supersymmetry then requires the lagrangian density for the spin-3/2 particle to be described by

$$\frac{1}{e} \mathcal{L}_{VS} = -\frac{1}{2} (\bar{\psi}_M \Gamma^{MNP} D_N \psi_P - m_{3/2} \bar{\psi}_M \Gamma^{MN} \psi_N), \quad (\text{A.2})$$

where we shall see how the parameter $m_{3/2}$ is related by supersymmetry to the cosmological constant. The presence of this ‘mass’ term does not mean that supersymmetry is broken; rather it is required in order to ensure that the gravitino/graviton action remains gauge invariant.

The combined gravitino-graviton lagrangian is invariant under the linearized supersymmetry transformations

$$\begin{aligned} \delta e_M^A &= -\frac{\kappa}{4} \bar{\psi}_M \Gamma^A \epsilon + \text{c.c.} \\ \delta \psi_M &= \frac{1}{\kappa} \left(D_M + \frac{1}{(n-2)} m_{3/2} \Gamma_M \right) \epsilon, \end{aligned} \quad (\text{A.3})$$

provided $m_{3/2}$ is related to Λ by

$$\Lambda = \frac{2(n-1)}{(n-2)} m_{3/2}^2. \quad (\text{A.4})$$

Notice that for any $n > 2$ this requires $\Lambda > 0$, which in our conventions corresponds to having anti-de Sitter space as the maximally-symmetric background solution. In 4D this reduces to the standard result $\Lambda_4 = 3m_{3/2}^2$ [29].

To put the spin-3/2 lagrangian into a form for which the general expressions for the Gilkey coefficients apply, we now use the gauge-averaging term

$$\frac{1}{e} L_{VS}^{gf} = -\frac{1}{2\xi_{3/2}} (\bar{\Gamma} \cdot \bar{\psi})(\not{D} + \gamma)(\Gamma \cdot \psi). \quad (\text{A.5})$$

After making the field redefinition $\psi_M \rightarrow \psi_M + A \Gamma_M \Gamma \cdot \psi$, we find that the following choices for A , ξ , and γ

$$A = \frac{1}{2-n}, \quad \xi_{3/2}^{-1} = \frac{2-n}{4}, \quad \gamma = \left(\frac{n}{2-n} \right) m_{3/2}, \quad (\text{A.6})$$

lead to the an expression for the vector-spinor lagrangian given by

$$\frac{1}{e} (\mathcal{L}_{VS} + \mathcal{L}_{VS}^{gf}) = -\frac{1}{2} \bar{\psi}_M (\not{D} + m_{3/2}) \psi^M. \quad (\text{A.7})$$

Following the analogous procedure in the main text, we obtain the result for the vector-spinor field in the presence of a cosmological constant:

$$\begin{aligned} \text{tr}_{VS}(a_0) &= \frac{n}{2} \mathcal{N}_{3/2} \\ \text{tr}_{VS}(a_1) &= n \mathcal{N}_{3/2} \left(\frac{1}{24} R - \frac{1}{2} m_{3/2}^2 \right) \\ \text{tr}_{VS}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7n}{8} \right) R_{MNPQ} R^{MNPQ} - n R_{MN} R^{MN} + \frac{5n}{8} R^2 + \frac{3n}{2} \square R \right] \\ &\quad + \frac{n \tilde{d} g_a^2}{12} C(\mathcal{R}_{3/2}) F_{MN}^a F_a^{MN} + \frac{n}{24} \mathcal{N}_{3/2} (-m_{3/2}^2 R + 6 m_{3/2}^4) \end{aligned} \quad (\text{A.8})$$

with $m_{3/2}^2$ defined by eq. (A.4).

The ghost action may be read from the supersymmetry transformation rules, from which we see that $\delta(\Gamma \cdot \psi) = \frac{1}{\kappa} [\not{D} + \frac{n}{n-2} m_{3/2}] \epsilon$, and so we find two bosonic, Faddeev-Popov spinor ghosts with the lagrangian

$$\frac{1}{e} \mathcal{L}_{LVFPgh} = -\bar{\omega}^i \left(\not{D} + \frac{n m_{3/2}}{n-2} \right) \omega_i. \quad (\text{A.9})$$

This has the same form as the spin-1/2 lagrangian, eq. (2.13), although with a Λ -dependent mass. In order to use this we require the following spin-1/2 results for the Gilkey-DeWitt coefficients quoted in the main text, generalized to include the fermion mass, m^2 , inside X :

$$\begin{aligned} \text{tr}_{1/2}(a_0) &= \frac{\mathcal{N}_{1/2}}{2} \\ \text{tr}_{1/2}(a_1) &= \frac{\mathcal{N}_{1/2}}{24} (R - 12m^2) \\ \text{tr}_{1/2}(a_2) &= \frac{\mathcal{N}_{1/2}}{360} \left[-\frac{7}{8} R_{MNPQ} R^{MNPQ} - R_{MN} R^{MN} + \frac{5}{8} (R - 12m^2)^2 + \frac{3}{2} \square R \right] \\ &\quad + \frac{\tilde{d} g_a^2}{12} C(\mathcal{R}_{1/2}) F_{MN}^a F_a^{MN}. \end{aligned} \quad (\text{A.10})$$

The Faddeev-Popov ghost result for $\text{tr}[a_k]$ is then obtained by multiplying these expressions by -2 , and specializing to the ‘mass’ $m = n m_{3/2}/(n-2)$.

The use of the operator $(\not{D} + \gamma)$ in the gauge-fixing lagrangian, eq. (A.5), leads to a bosonic, Nielsen-Kallosh ghost. Rewriting γ in terms of $m_{3/2}$, we see that the

Nielsen-Kallosh ghost has the lagrangian

$$\frac{1}{e} \mathcal{L}_{LVNKgh} = -\bar{\omega} \left(\not{D} - \frac{n m_{3/2}}{n-2} \right) \omega. \quad (\text{A.11})$$

This ghost therefore contributes -1 times the spin-1/2 result to $\text{tr}[a_k]$, with $m = -nm_{3/2}/(n-2)$.

Adding the vector-spinor result together with its associated ghosts, we obtain the following contribution to $\text{tr}[a_k]$ by physical spin-3/2 states in the presence of a cosmological constant:

$$\begin{aligned} \text{tr}_{3/2}(a_0) &= \frac{\mathcal{N}_{3/2}}{2}(n-3) \\ \text{tr}_{3/2}(a_1) &= \frac{\mathcal{N}_{3/2}}{24} \left((n-3)R - \frac{6n(n^2-7n+4)}{(n-1)(n-2)}\Lambda \right) \\ \text{tr}_{3/2}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7}{8}(n-3) \right) R_{MNPQ}R^{MNPQ} - (n-3)R_{MN}R^{MN} \right. \\ &\quad + \frac{5}{8}(n-3)R^2 + \frac{3}{2}(n-3)\square R - \frac{15n(n^2-7n+4)\Lambda R}{2(n-1)(n-2)} \\ &\quad \left. + \frac{45n(n^4-11n^3+24n^2-32n+16)\Lambda^2}{2(n-1)^2(n-2)^2} \right] \\ &\quad + \frac{\tilde{d}g_a^2}{12}(n-3)C(\mathcal{R}_{3/2})F_{MN}^aF_a^{MN}. \end{aligned} \quad (\text{A.12})$$

Massive Gravitino

This section follows closely the procedure outlined in the massive spin-3/2 section of the main text. Starting from eq. (2.62), which was our ansatz for a massive spin-3/2 lagrangian, we again find that this lagrangian can be made invariant under the supersymmetry transformations

$$\delta\psi_M = \frac{1}{\kappa}D_M\epsilon + \mu\Gamma_M\epsilon \quad \text{and} \quad \delta\chi = f\epsilon. \quad (\text{A.13})$$

In this case, however, f is given by

$$f^2 = (n-1)(n-2)\mu^2 - \frac{\Lambda}{2\kappa^2}, \quad (\text{A.14})$$

while all other equations in eq. (2.64) remain unchanged. The dependence of f on Λ is required in order to cancel the variation of the Λ term in the Einstein-Hilbert action.

Again, following the procedure of the main text, we add a gauge-fixing term, eq. (2.67), and perform a field redefinition, eq. (2.70), in order to put the lagrangian

into the form

$$\frac{1}{e} (\mathcal{L}_{mVS} + \mathcal{L}_{mVS}^{gf}) = -\bar{\psi}'_M (\not{D} + m'_{3/2}) \psi'^M - \bar{\chi}' (\not{D} + m'_{1/2}) \chi'. \quad (\text{A.15})$$

The parameters in the gauge-fixing lagrangian and in the field redefinitions can be written in terms of M and \hat{M} , defined as

$$M = (n-2)\kappa\mu \quad \text{and} \quad \hat{M} = \sqrt{M^2 + \frac{2\Lambda}{n-2}}. \quad (\text{A.16})$$

With these definitions, we find

$$\begin{aligned} A &= \sqrt{1 + \beta^2}, & B &= -\frac{\beta}{2}\sqrt{n-2}, & C &= -\frac{1}{2}, & D &= 0, \\ \alpha &= -\frac{1}{2}\sqrt{(n-2)(1 + \beta^2)}, & \beta &= \left[\frac{1}{2} \left(\frac{n}{n-2} \right) \frac{M}{\hat{M}} - \frac{1}{2} \right]^{1/2}, \\ m'_{1/2} &= -\gamma = \hat{M}, & m'_{3/2} &= M. \end{aligned} \quad (\text{A.17})$$

For the case $\Lambda = 0$, these expressions reduce to those given in eq. (2.71). There is a possible subtlety in the above solution, which comes about because of our simplifying assumption to take all free parameters to be real. We see that for certain choices of M and Λ , it's possible that some of the parameters will be imaginary. However, in the situations for which our results apply we expect that $M \gg |\Lambda|$, and so in these cases this problem will not arise.

From the gauge-fixing condition, we see that there are two Faddeev-Popov ghosts, each with mass \hat{M} , and one Nielsen-Kallosh ghost, with mass $-\hat{M}$. The one loop effective action for the ghosts is thus given by

$$\begin{aligned} i\Sigma_{1/2} &= \frac{1}{4} \text{Tr} \log \left(\hat{M}^2 - \not{D}^2 \right) \\ &= \frac{1}{4} \text{Tr} \log \left(M^2 + \frac{2\Lambda}{n-2} - \not{D}^2 \right). \end{aligned} \quad (\text{A.18})$$

As usual, we factor the M^2 dependence out of our definition of X , and so obtain

$$X = -\frac{1}{4}R + \frac{i}{2}\Gamma^{AB}F_{AB}^a t_a + \frac{2\Lambda}{n-2}. \quad (\text{A.19})$$

The contribution to the Gilkey coefficients coming from the three ghosts is thus obtained by multiplying eq. (A.10) by -3 , with $m^2 = 2\Lambda/(n-2)$. Similarly, the Goldstone fermion contribution is also given by eq. (A.10), again with $m^2 = 2\Lambda/(n-2)$. The contribution from the vector spinor is unchanged from the massless case considered in the main text, and so its Gilkey coefficients are given by eq. (2.58).

Summing these results, we arrive at the expression for a massive gravitino in a background spacetime having nonzero cosmological constant:

$$\begin{aligned}
\text{tr}_{m3/2}(a_0) &= \frac{\mathcal{N}_{3/2}}{2}(n-2) \\
\text{tr}_{m3/2}(a_1) &= \frac{\mathcal{N}_{3/2}}{24} \left((n-2)R + \frac{48\Lambda}{n-2} \right) \\
\text{tr}_{m3/2}(a_2) &= \frac{\mathcal{N}_{3/2}}{360} \left[\left(30 - \frac{7}{8}(n-2) \right) R_{MNPQ}R^{MNPQ} - (n-2)R_{MN}R^{MN} \right. \\
&\quad \left. + \frac{5}{8}(n-2)R^2 + \frac{3}{2}(n-2)\square R + \frac{60\Lambda R}{(n-2)} - \frac{720\Lambda^2}{(n-2)^2} \right] \\
&\quad + \frac{g_a^2}{12}(n-2)\tilde{d}C(R_{3/2})F_{MN}^aF_a^{MN}. \tag{A.20}
\end{aligned}$$

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